SUBJECT: MATHEMATICS

GRADE 12

2026 WINTER CLASSES

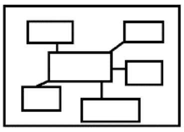
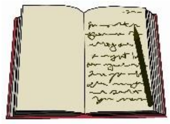
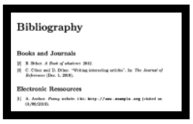
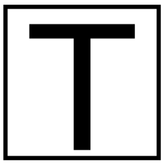
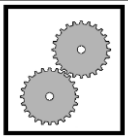
TEACHER AND LEARNER CONTENT MANUAL

Topic(s)

FUNCTIONS AND GRAPHS

A Division of Marematlou Group Holdings

ICON DESCRIPTION

 MIND MAP	 EXAMINATION GUIDELINE	 CONTENTS	 ACTIVITIES
 BIBLIOGRAPHY	 TERMINOLOGY	 WORKED EXAMPLES	 STEPS

CONTENTS **PAGE**

FUNCTIONS AND GRAPHS

Overview:

Revise the effect of the parameters a and q and investigate the effect of p on the graphs of the functions defined by: 1.1. $y = f(x) = a(x+p)^2 + q$ 1.2. $y = f(x) = \frac{a}{x+p} + q$ 1.3. $y = f(x) = a.b^{x+p} + q$ where $b > 0, b \neq 1$	Investigate numerically the average gradient between two points on a curve and develop an intuitive understanding of the concept of the gradient of a curve at a point.	1. Definition of a *function*. 2. General concept of the *inverse of a function* and how the domain of the function may need to be restricted (in order to obtain a one-to-one function) to ensure that the inverse is a function. 3. Determine and sketch graphs of the inverses of the functions defined by $y = ax + q$; $y = ax^2$ $y = b^x; (b > 0, b \neq 1)$ Focus on the following characteristics: domain and range, intercepts with the axes, turning points, minima, maxima, asymptotes (horizontal and vertical), shape

(SOURCE: CURRICULUM AND ASSESSMENT POLICY STATEMENT(CAPS)SENIOR PHASE GRADES (10 – 12) MATHEMATICS)

Important terminilogy

Domain:	the set of possible x-values	For all functions
Range:	the set of possible y-values	For all functions
Axis of symmetry:	an imaginary line that divides a graph into two mirror images of each other.	See the hyperbola and parabola
Maximum:	the highest possible y-value of a function.	See the parabola
Minimum:	the lowest possible y-value of a function.	See the parabola
Asymptote:	an imaginary line that a graph approaches but never touches.	See the hyperbola and exponential function
Turning point:	The point at which a graph reaches its maximum or minimum value and changes direction.	See the parabola

The concepting of increasing and decresing in functions: all functions

- The function is INCREASING when the value of y increases as x is increasing from left to right
  - **THE GRAPH GOES UP**

- The function is DECREASING when the value of y decreases as x is increasing from left to right
  - **THE GRAPH GOES DOWN**

SECTION 1: LINEAR FUNCTION (STRAIGHT LINE)

The graph of $y = mx + c$

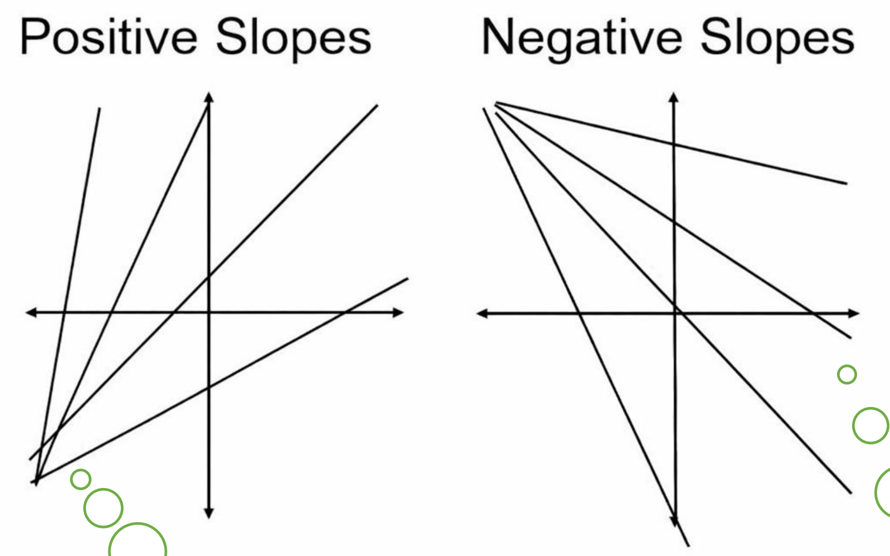
Standard form of linear function

WHERE

$m = gradient$

When $m > 0$ ***(gradient is positive)*** and
$m < 0$ ***(gradient is negative)***

Domain: $x \in R$
Range: $y \in R$
Shape



When $m > 0$

3. **The gradient is positive**
4. **The function is increasing**

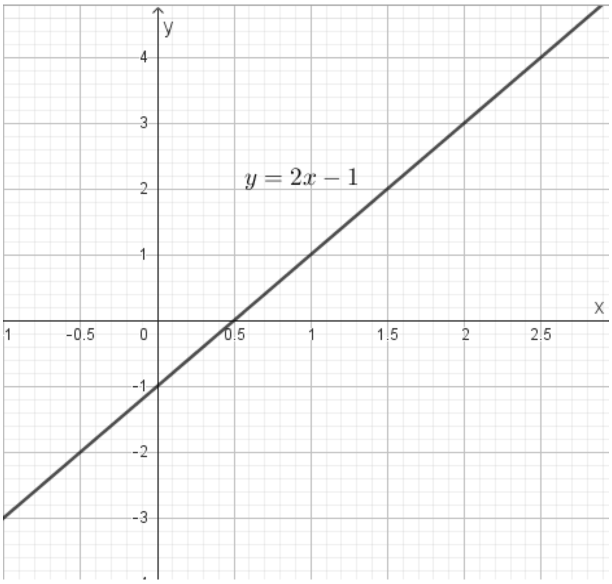
When $m < 0$

1. **The gradient is negative**
2. **The function is**

Example 1

Sketch the graph of $y = 2x - 1$ and determine the domain and range of the function, and state if the function is increasing or decreasing.

Solution



$\boldsymbol{x - intercept: ley\ y = o}$

$0 = 2x - 1$

$-2x = -1$

$x = \frac{1}{2}$

$\therefore \left(\frac{1}{2}; 0\right)$

$y - intercept: let\ x = 0$

$y = 2(0) - 1$

$y = -1$

$\therefore (0; -1)$

Domain: $x \in R$

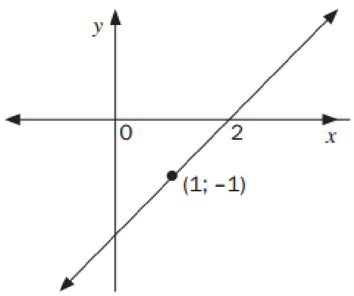
Range:$y \in R$

The function is increasing

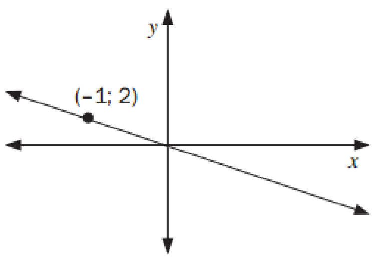
Example 2

Determine the equations of the following graphs:

1.



2.



Solutions

1.

$a = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-1 - 0}{1 - 2}$

$a = 1$

$\therefore y = 1x + c$

$0 = 1(2) + c$

$c = -2$

$y = x - 2$

2.

$a = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{2 - 0}{-1 - 0}$

$a = -2$

$\therefore y = -2x + c$

$0 = -2(0) + c$

$c = 0$

$y = -2x$

SECTION 2: HYPERBOLIC FUNCTIONS(HYPERBOLA)

The graph of $y = \frac{a}{x+p} + q$

Standard form of hyperbola

take note that $\boldsymbol{y = \frac{2}{x-2} + 1}$

$$\boldsymbol{= \frac{2}{x + (-2)} + 1}$$

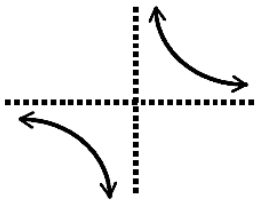
The equations of asymptotes are $x = -p$ ***(vertical asymptote)*** and
$y = q$ ***(horizontal asymptote)***

Domain: $x \in R, x \neq -p$

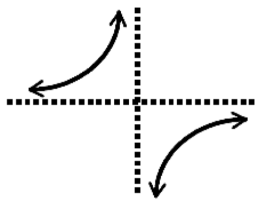
Range: $y \in R, y \neq q$

Shape

If $a > 0$ then the graph decreases for all $x < 0$ or $x > 0$.



If $a < 0$ then the graph increases for all $x < 0$ or $x > 0$.



The equations of the axis of symmetry

The hyperbola has two equations of symmetry

$\boldsymbol{m = 1}$	$\boldsymbol{m = -1}$
$\boldsymbol{y = x + c}$	$\boldsymbol{y = -x + c}$

N.B the equations of the axis of symmetry of the hyperbola passes through the point of intersection of asymptotes $(-p; q)$

In general, for the hyperbola, the equations of the axis of symmetry are given by the following formulae:

$\boldsymbol{m = 1}$	$\boldsymbol{m = -1}$
$\boldsymbol{y = (x + p) + q}$ $\boldsymbol{\therefore y = x + p + q}$	$\boldsymbol{y = -(x + p) + q}$ $\boldsymbol{\therefore y = -x - p + q}$

N.B Ensure that the hyperbola is in standard form before applying the formula

Example 1

Sketch the graph of $y = \frac{10}{x+2} - 3$, write down the domain and range of the function, and write down the equations of asymptotes. state whether the function is increasing or decreasing. Lastly, determine the equation of the asymptote with a positive gradient.

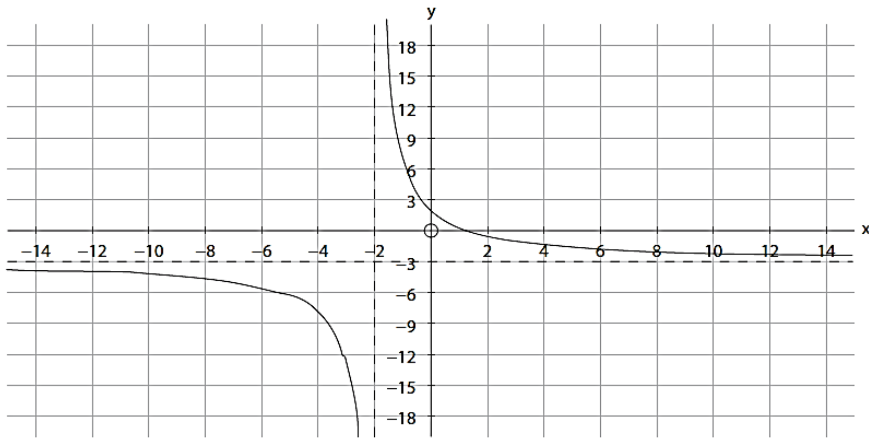
Solutions

The asymptotes are $x = -2$ and $y - 3$. (These are read from the equation).

x-intercept: Let $y = 0$:

$0 = \frac{10}{x+2} - 3 \therefore 10 = 3(x+2) \quad \therefore 3x = 4 \quad \therefore x = \frac{4}{3}$

y-intercept: Let $x = 0$: $y = \frac{10}{0+2} - 3 = 5 - 3 = 2$



graph not drawn to scale

Domain: $x \in R, x \neq -2$

Range: $y \in R, y \neq -3$

The function is decreasing

<u>Equation of the axis of symmetry with positive gradient:</u>

$y = x + c$

$-3 = -2 + c$

$c = -1$

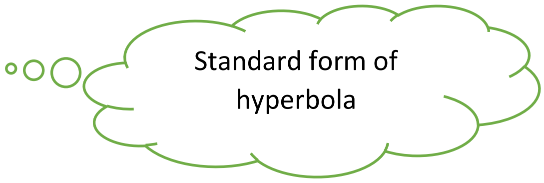
$\therefore y = x - 1$

Example 2

Sketch the graph of $y = -1 - \frac{8}{x-4}$, and write down the domain and range of the function. And determine the equation of the axis of symmetry with a negative gradient.

Solution

This equation can also be written as $y = \frac{-8}{x-4} - 1$.

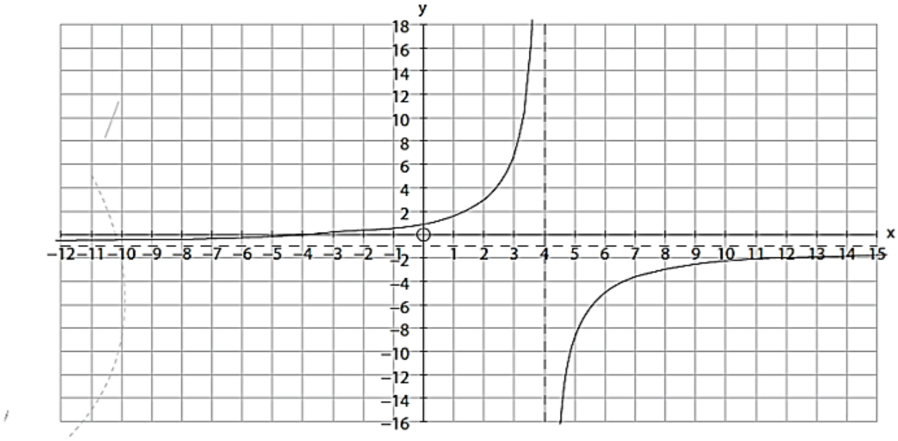


Asymptotes: $x = 4$ and $y = -1$

y-intercept: Let $x = 0$: $y = -1 - \frac{8}{0-4} = -1 + 2 = 1$

x-intercept: Let $y = 0$: $0 = -1 - \frac{8}{x-4}$ $\quad \therefore (x-4) = -8 \quad \therefore x = -4$

The hyperbola will look as follows:



graph not drawn to scale

Domain: $x \in R, x \neq 4$
Range: $y \in R, y \neq -1$

The function is increasing

<u>Equation of the axis of symmetry with positive gradient:</u>
$y = -x + c$
$-1 = -(4) + c$
$c = 3$
$\therefore y = -x + 3$

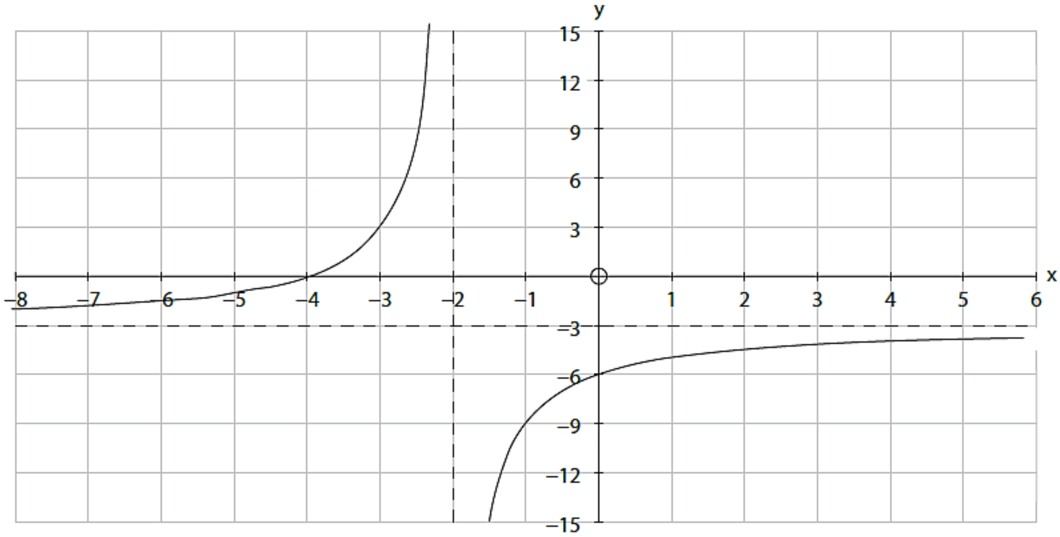
Example 3

Determining the equation of a hyperbola when the graph is given

Step 1: The position of the asymptotes give us the value(s) of p and q in $y = \frac{a}{x+p} + q$.

Step 2: To find the value of a, we substitute any point on the graph into the equation.

EXAMPLE: The graph of $y = \frac{a}{x+p} + q$ is sketched below. Determine the value(s) of a, p and q.



From the graph we see that, $p = 2$ and $q = -3$.

The equation becomes: $y = \frac{a}{x+2} - 3$.

Substituting (0; –6) or any other point on the graph into the equation gives:

$-6 = \frac{a}{0+2} - 3 \quad \therefore -12 = a - 6 \quad \therefore a = -6$

$$\therefore y = -\frac{6}{x+2} - 3$$

N.B the function is increasing

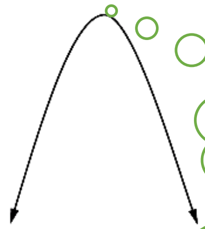
SECTION 3: QUADRATIC FUNCTION(PARABOLA)

The graph of $y = a(x+p)^2 + q$

If a is positive, i.e. a > 0, then the shape of the graph is ☺.

Minimum turning point

If a is negative, i.e. a < 0, then the shape of the graph is ☹.



Maximum turning point

The graph has the axis of symmetry at $x = -p$

The graph has the turning point by $(-p\ ;\ q)$

Domain: $x \in R$

Range: $y \geq q)$ 😊 **(WHEN a > 0)**
or
$y \leq q$ 😔 **(WHEN a < 0)**

N.B

1. **q is the minimum of the parabola when $a > 0$**

2. **q is the maximum of the parabola when $a < 0$**

N.B The parabola changes from increasing to decreasing or decreasing to increasing at the turning point.

<u>when a > 0</u> 1. $\boldsymbol{The\ graph\ increases\ for: x > -p}$ 2. $\boldsymbol{The\ graph\ decrease\ for: x < -p}$
<u>when a < 0</u> 1. $\boldsymbol{The\ graph\ increases\ for: x < -p}$ 2. $\boldsymbol{The\ graph\ decrease\ for: x > -p}$

The quadratic function can also be represented in the form:

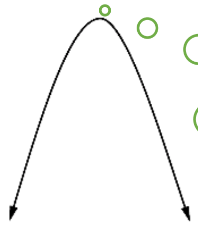
$y = f(x) = ax^2 + bx + c$

Standard form of parabola

If a is positive, i.e. a > 0, then the shape of the graph is ☺.

Minimum turning point

If a is negative, i.e. a < 0, then the shape of the graph is ☹.



Maximum turning point

The graph has the axis of symmetry at $x = -\frac{b}{2a}$

The graph has the turning point by $(-\frac{b}{2a}\ ;\ f(-\frac{b}{2a}))$

Domain: $x \in R$

Range: $y \geq f(-\frac{b}{2a})$ 😊 **(WHEN a > 0)**

or

$y \leq f(-\frac{b}{2a})$ 😔 **(WHEN a < 0)**

N.B

1. $f\left(-\frac{b}{2a}\right)$ is the minimum of the parabola when $a > 0$

2. $f\left(-\frac{b}{2a}\right)$ is the maximum of the parabola when $a < 0$

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Example 1

Sketch the graph of $f(x) = x^2 - 5x - 6$, clearly showing intercepts with the axes and the turning point. Also give the interval where the function is increasing and where the function is decreasing. Lastly, determine the domain and range of of $f(x) = x^2 - 5x - 6$.

Solutions

Sketch the graph of $f(x) = x^2 - 5x - 6$

1. ***y*-intercept**

   $f(0) = -6$

   Therefore the co-ordinates of the y-intercept are $(0; -6)$

2. ***x*-intercept**

   $x2 - 5x - 6 = 0$

   $(x - 6)(x + 1) = 0$

   $x = 6$ or $x = -1$

   $(6; 0)$ and $(-1; 0)$

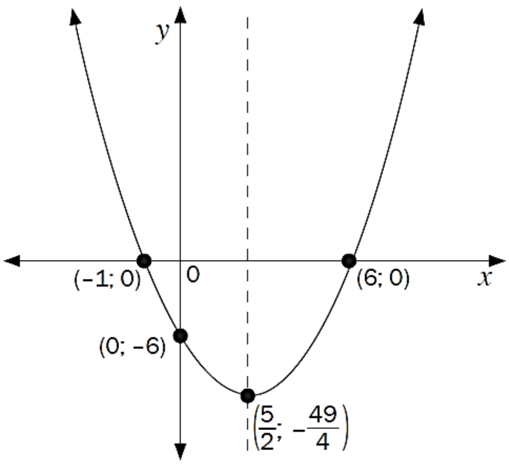
3. **Axis of symmetry**

   $$x = \frac{-b}{2a}$$
   $$= \frac{-(-5)}{2(1)}$$
   $$= \frac{5}{2}$$

4. **Turning point**

   $$f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) - 6$$
   $$= -12\frac{1}{4}$$
   $$\therefore TP\left(\frac{5}{2}; -12\frac{1}{4}\right)$$

5. **Sketch graph**



6. The interval where the graph is increasing

$x > \frac{5}{2}$

The interval where the graph is decreasing

$x < \frac{5}{2}$

7. Domain: $x \in R$

Range:$y \geq -\frac{49}{4}$

Determining the equation of the quadratic function

Given the x-intercepts and one point	Given the turning point and one point
• Use the formula: $y = a(x - x_1)(x - x_2)$. • Substitute the values of the x-intercepts. • Substitute the given point which is not the x-intercept. • Solve for a. • Write the equation in the form $f(x) = ax^2 + bx + c$.	• Use the formula: $y = a(x + p)^2 + q$. • Substitute the co-ordinates of the turning point $(p; q)$. • Substitute the given point. • Solve for a. • Write the equation in the form $y = a(x + p)^2 + q$ or $f(x) = ax^2 + bx + c$ depending on the instruction in the question.

Given the co-ordinates of three points on the parabola
• Use the formula: $y = ax^2 + bx + c$. • One of the given point is the y-intercept, therefore c is given, so substitute its value. • Substitute the co-ordinates of the other two points into $y = ax^2 + bx + c$. • Solve the two equations simultaneously for a and b.

SOURCE:Mind The Gap Mathematics Study Guide

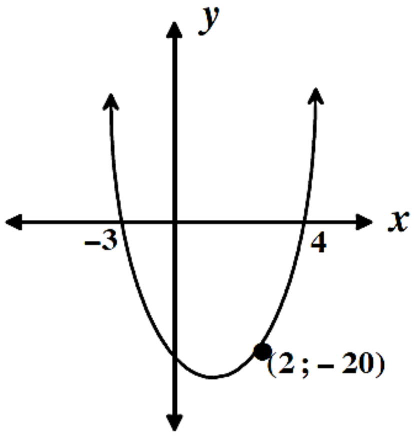
Example 2

Determine the equation of the parabola in the form $f(x) = ax^2 + bx + c$.

$y = a(x - x_1)(x - x_2)$

$\therefore y = a(x - (-3))(x - 4)$

$\therefore y = a(x + 3)(x - 4)$



Substitute $(2\,;-20)$ to find the value of a.

$-20 = a(2 + 3)(2 - 4)$

$\therefore -20 = a(5)(-2)$

$\therefore -20 = -10a$

$\therefore a = 2$

$\therefore y = 2(x + 3)(x - 4)$

$\therefore y = 2(x^2 - x - 12)$

$\therefore f(x) = 2x^2 - 2x - 24$

Example 3

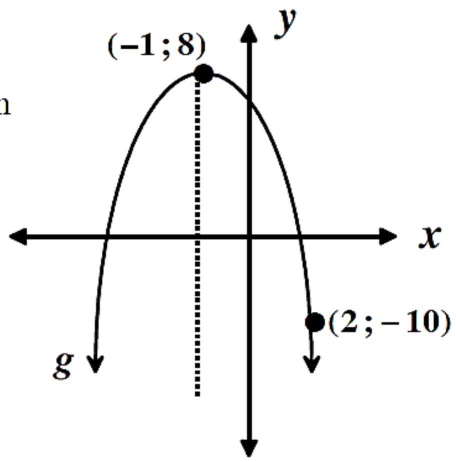
Determine the equation of g in the form $y = ax^2 + bx + c$

The axis of symmetry is given by $x = -1$.

$\therefore x + 1 = 0$

From the work on parabolas, it is clear that the expression $x + 1$ is in the brackets of the equation for a parabola.

Also, the value of q is 8 (the y-value of the turning point).



$\therefore y = a(x + 1)^2 + 8$

Now substitute the point $(2\,;-10)$ which lies on the graph of the parabola.

$-10 = a(2 + 1)^2 + 8$

$\therefore -18 = a(3)^2$

$\therefore -18 = 9a$

$\therefore a = -2$

$\therefore y = -2(x + 1)^2 + 8$

$\therefore y = -2(x^2 + 2x + 1) + 8$

$\therefore y = -2x^2 - 4x - 2 + 8$

$\therefore y = -2x^2 - 4x + 6$

SECTION 4: EXPONENTIAL FUNCTION

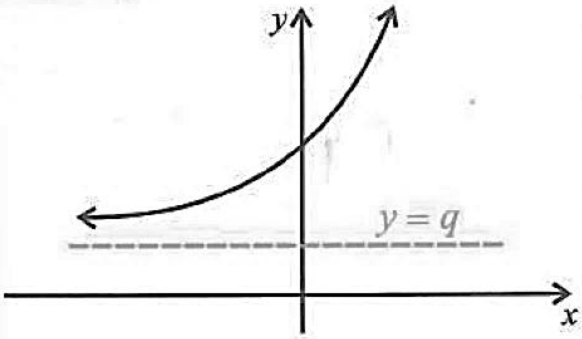
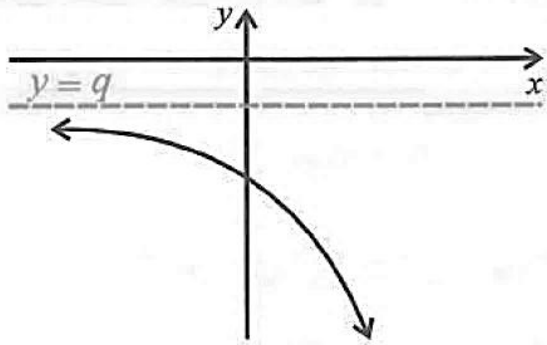
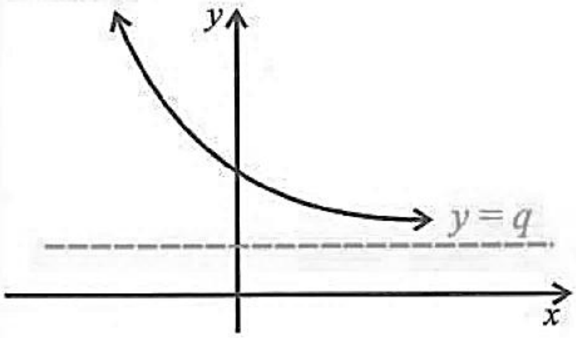
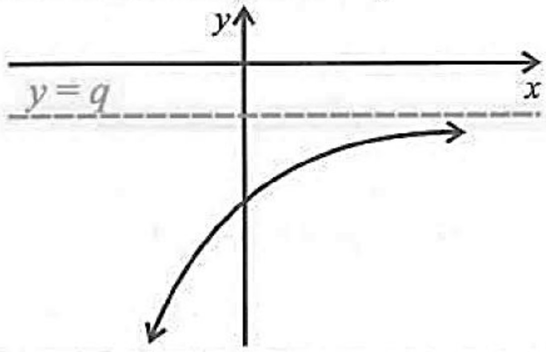
Standard form of exponential function

The graph of $\boldsymbol{y = a.b^{x+p} + q}$ **where** $\boldsymbol{b > 0}$ **and** $\boldsymbol{b \neq 1}$

The equation of an asymptote is $y = q$ ***(horizontal asymptote)***

Domain: $x \in R$

Range: $y > q$ *[if a > 0]* or $y < q$ *[if a < 0]*

$a > 0$ and $b > 1$	$a < 0$ and $b > 1$
The graph lies above the horizontal asymptote and is an increasing function. 	The graph lies below the horizontal asymptote and is a decreasing function. 
$a > 0$ and $0 < b < 1$	$a < 0$ and $0 < b < 1$
The graph lies above the horizontal asymptote and is a decreasing function. 	The graph lies below the horizontal asymptote and is an increasing function. 

(Source: MATHS MADE EASY – A comprehensive guide to Grade 12 Mathematics)

Example 1

Sketch the graph of $y = 2\left(\frac{1}{2}\right)^{x+1} - 4$

Horizontal asymptote: $y = -4$

x-intercept: let $y = 0$:

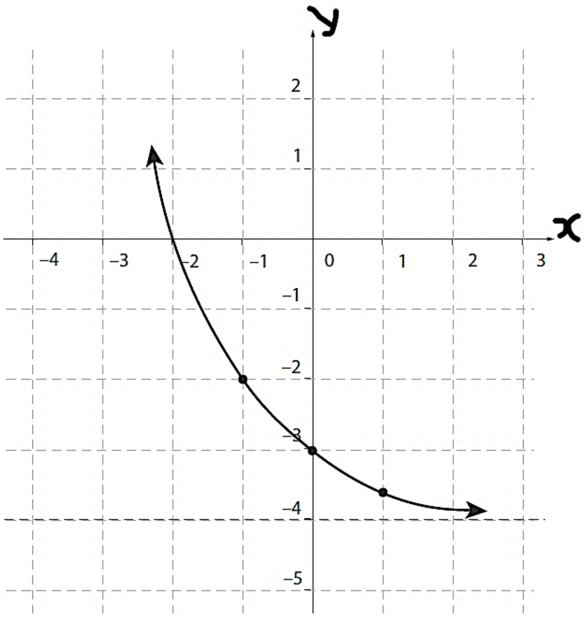
$0 = 2\left(\frac{1}{2}\right)^{x+1} - 4 \therefore \left(\frac{1}{2}\right)^{x+1} = 2$

$\left(\frac{1}{2}\right)^{x+1} = \left(\frac{1}{2}\right)^{-1}$

$\therefore x + 1 = -1 \quad \therefore x = -2$

y-intercept: let x = 0

y = - 3



N.B the function is decreasing

Example 2

Let us look at a second example, $y = 3 \cdot (3)^{x-2} + 1$

Horizontal asymptote: $y = 1$

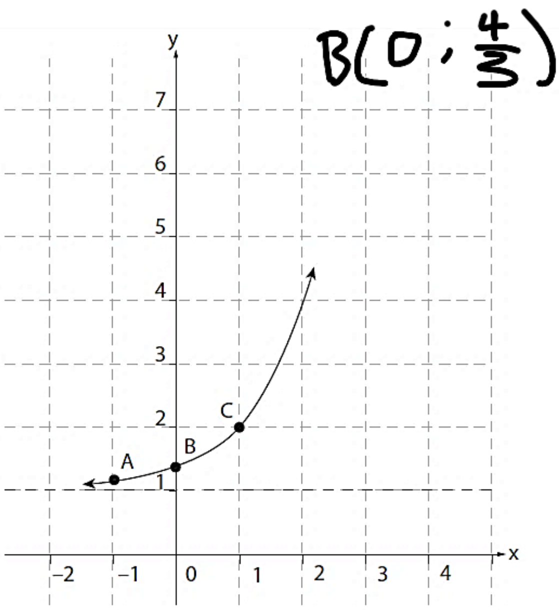
x-intercept: $0 = 3 \cdot (3)^{x-2} + 1$

$\therefore 3(3)^{x-2} = -1$

$\therefore$ No x-intercept

y-intercept: let x = 0

$y = \frac{4}{3}$

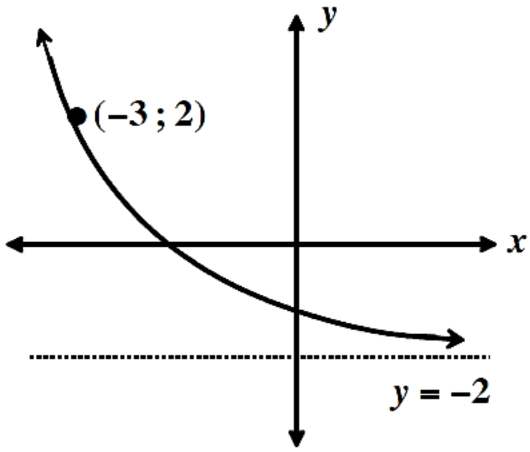


N.B the function is inccreasing

Determining the equation of exponential function

Example 3

Determine the equation of $g(x) = b^{x+1} + q$



Horizontal asymptote: $y = -2$

Therefore the value of q is -2

$\therefore y = b^{x+1} - 2$

Now substitute the point $(-3\,;2)$ into the equation to get b.

$2 = b^{-3+1} - 2$

$\therefore 4 = b^{-2}$

$\therefore 4 = \dfrac{1}{b^2}$

$\therefore 4b^2 = 1$

$\therefore 4b^2 - 1 = 0$

$\therefore (2b+1)(2b-1) = 0$

$\therefore b = -\dfrac{1}{2}$ or $b = \dfrac{1}{2}$

But $b \neq -\dfrac{1}{2}$

$\therefore b = \dfrac{1}{2}$

Note:
Since $b > 0$, you could have solved the equation $4b^2 = 1$ as follows:

$4b^2 = 1$

$\therefore b^2 = \dfrac{1}{4}$

$\therefore b = \dfrac{1}{2}$

Therefore the equation is: $g(x) = \left(\dfrac{1}{2}\right)^{x+1} - 2$

TRANSFORMATION OF FUNCTIONS

REFLECTIONS AND TRANSLATIONS

Given $f(x) = \frac{2}{x+1} - 3$	Given $f(x) = 2.3^{x-2} + 4$	Given $f(x) = x^2 + 5x + 6$
a. The graph of $g(x)$ is obtained by shifting the graph of $f(x)$ 2 units up and 3 units to left. Determine the equation of $g(x)$. **Solution** $g(x) = f(x+3) + 2$ $= \frac{2}{x+3+1} - 3 + 2$ $= \frac{2}{x+4} - 1$ b. The graph of $h(x)$ is obtained by reflecting the graph of $f(x)$ in the $x - axis$. Determine the equation of $h(x)$. **Solution** $h(x) = -f(x)$ $= -\left(\frac{2}{x+1} - 3\right)$ $= -\frac{2}{x+1} + 3$ c. The graph of $m(x)$ is obtained by reflecting the graph of $f(x)$ in the $y - axis$. Determine the equation of $m(x)$. **Solution** $m(x) = f(-x)$ $= \frac{2}{-x+1} - 3$ $= \frac{2}{-(x-1)} - 3$ $= -\frac{2}{x-1} - 3$	a. The graph of $g(x)$ is obtained by shifting the graph of $f(x)$ 2 units up and 3 units to left. Determine the equation of $g(x)$. **Solution** $g(x) = f(x+3) + 2$ $= 2.3^{x+3-2} + 4 + 2$ $= 2.3^{x+1} + 6$ b. The graph of $h(x)$ is obtained by reflecting the graph of $f(x)$ in the $x - axis$. Determine the equation of $h(x)$. **Solution** $h(x) = -f(x)$ $= -(2.3^{x-2} + 4)$ $= -2.3^{x-2} - 4$ c. The graph of $m(x)$ is obtained by reflecting the graph of $f(x)$ in the $y - axis$. Determine the equation of $m(x)$. **Solution** $m(x) = f(-x)$ $= 2.3^{-x-2} + 4$ $= 2.3^{-(x+2)} + 4$ $= 2.\left(\frac{1}{3}\right)^{x+2} + 4$	a. The graph of $g(x)$ is obtained by shifting the graph of $f(x)$ 2 units down and 3 units to right. Determine the equation of $g(x)$. **Solution** $g(x) = f(x-3) - 2$ $= (x-3)^2 + 5(x-3) + 6 - 2$ $= x^2 - 6x + 9 + 5x - 15 + 4$ $= x^2 - x - 2$ b. The graph of $h(x)$ is obtained by reflecting the graph of $f(x)$ in the $x - axis$. Determine the equation of $h(x)$. **Solution** $h(x) = -f(x)$ $= -(x^2 + 5x + 6)$ $= -x^2 - 5x - 6$ c. The graph of $m(x)$ is obtained by reflecting the graph of $f(x)$ in the $y - axis$. Determine the equation of $m(x)$. **Solution** $m(x) = f(-x)$ $= (-x)^2 + 5(-x) + 6$ $= x^2 - 5x + 6$

SECTION 5: INVERSE FUNCTIONS

A function f is defined as a relationship between values, where each input value maps to one output value. In other words, for an equation to be called a function, there can only be one y-value for a particular x-value.

There are two types of functions:

1. One-to-One Functions
2. Many-to-One Functions

ONE-TO-ONE FUNCTIONS

A one-to-one function is a function where there is a single y-value for a particular x-value.

MANY-TO-ONE FUNCTIONS

A function cannot have more than one y-value for each x-value. However, a function can have more than one x-value for a particular y-value. These are known as many-to-one functions.

VERTICAL LINE TEST

1.1.1 To test if a graph is a function, use the vertical line test. If a vertical line (a line parallel to the $\boldsymbol{y}$-axis) touches the graph more than once at any point, the graph is not a function. You don't have to draw a line; just hold a ruler parallel to the y-axis and move it along the graph. If the ruler touches the graph more than once for a single $\boldsymbol{x}$-value anywhere on the graph, then the graph is not a function. In the case of the graph not being a function, it is said to be a relation.

1.1.2 HORIZONTAL LINE TEST

If a graph passes the vertical line test, it is a function. The horizontal line test can be used to determine what type of function the graph represents.

If a horizontal line (a line parallel to the x-axis) is drawn and moved along the graph and it touches the graph more than once at any point, it is a many-to-one function (many x-values to a single y-value). Otherwise, it is a one-to-one function.

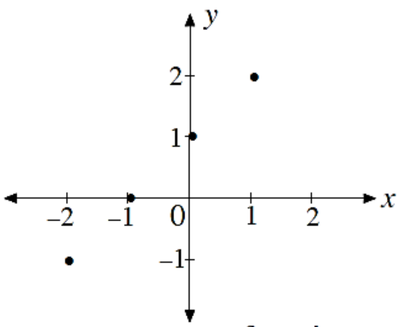
The horizontal line test does not determine whether a graph is a function, only the type of function. The vertical line test must always be used first. Only if this test shows that the graph is a function can the horizontal line test be used to determine the type of function. If a graph is not a function it is known as a relation.

(Source: MATHS MADE EASY – A comprehensive guide to Grade 12 Mathematics)

Examples

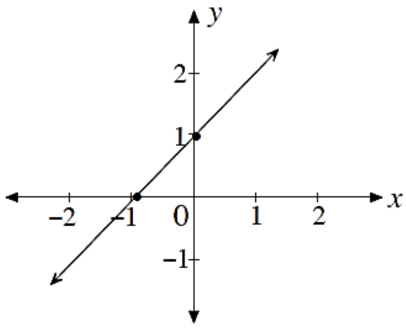
Determine whether the following relations are functions or not. If the graph is a function, determine whether the function is one-to-one or many-to-one.

(a)



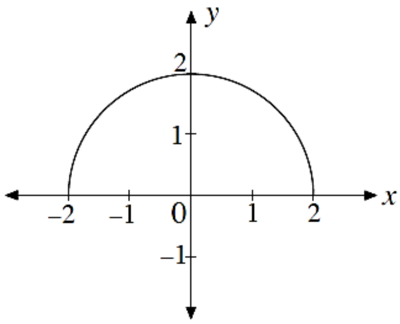
one-to-one function

(b)



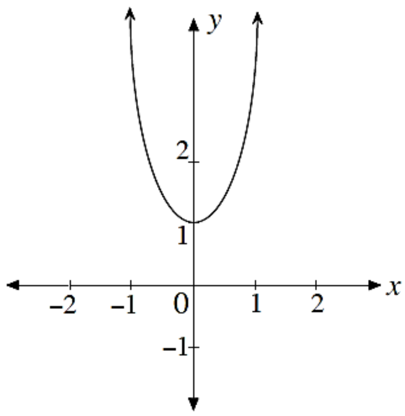
one-to-one function

(c)



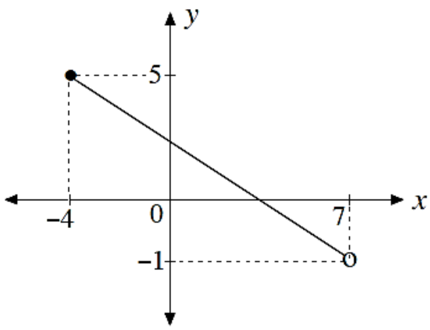
many-to-one function

(d)



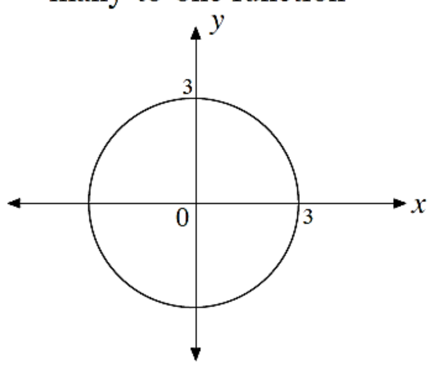
many-to-one function

(e)



one-to-one function

(f)



not a function

(Source: Mind Action series Mathematics12 textbook and Workbook)

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Inverse Function

- The inverse of a function takes the y-values (range) of the function to the corresponding x-values (domain) and vice versa. Therefore the x and y values are interchanged.
- The function is reflected along the line $y = x$ to form the inverse.
- The notation for the inverse of a function is f^{-1}.

- N.B The domain of the inverse is the range of the function and the range of the inverse is the domain of the function.
- When the function is increasing, its inverse also increases. When the function decreases, its inverse will also decrease.

Inverse function : Linear ($y = ax + q$)

Example 1

Given $f(x) = 2x + 6$.

1. Determine $f^{-1}(x)$
2. Sketch the graphs of $f(x)$, $f^{-1}(x)$ and $y = x$ on the same set of axis

Solutions

1. In order to find the inverse of a function, there are two steps:

STEP 1: Swap the x and y

$y = 2x + 6$

becomes $x = 2y + 6$

We then rewrite the equation to make y the subject of the formula. Therefore,

STEP 2: make y the subject of the formula

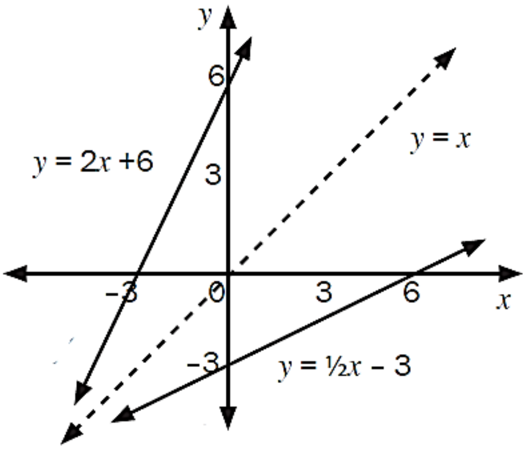
$x = 2y + 6$

$x - 6 = 2y$

So $y = \frac{1}{2}x - 3$

We can say that the inverse function $f^{-1}(x) = \frac{1}{2}x - 3$

2.



Inverse function : Quadratic ($y = ax^2$)

Example 2

a) Sketch $f(x) = 2x^2$

b) Determine the inverse of $f(x)$

c) Sketch $f^{-1}(x)$ and $y = x$ on the same axes as $f(x)$

a) & c)

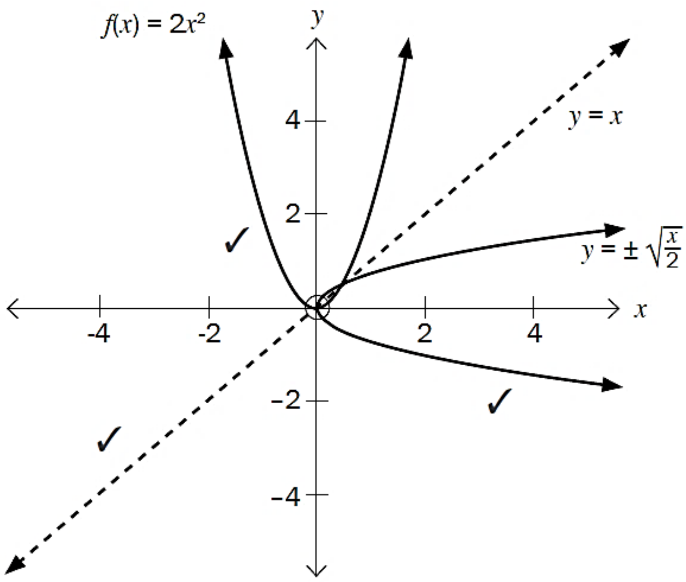
Solution

1. b) $y = 2x^2$

$x = 2y^2$ ✓

$y = \pm\sqrt{\frac{x}{2}}$ ✓

- This is not a function.
- Check it with a vertical line test. There are two y-values for one x-value.
- Not all inverses of functions are also functions. Some inverses of functions are relations.
- If an inverse is not a function, then we can restrict the **domain** of the **function** in order for the inverse to be a function.



- To make the inverse a function, we need to choose a set of x-values in the function and work only with those. We call this **'restricting the domain'**.
- A one to one function has an inverse that is a function

  Example: $y = 3x + 4$ is a one to one function. For every x value there is one and only one y value

- A many to one function has an inverse that is not a function. However, we can restrict the domain of the function to make its inverse a function.

  Example: $y = 2x^2$ is a many to one function. For two or many x values there is one y value. (if $x = 2$, then $y = 8$.
  If $x = -2$, then $y = 8$). Therefore, its inverse$y = \pm\sqrt{\frac{x}{2}}$, is not a function.
- To check for a function, draw a vertical line. If any vertical line cuts the graph in only one place, the graph is a function.
  If any vertical line cuts the graph in more than one place, then the graph is not a function.
- To check for a one-to-one function, draw a horizontal line. If any horizontal line cuts the graph in only one place, the graph is a one-to-one function. If any horizontal line cuts the graph in more than one place, then the graph is a many-to-one function.

Example 3

Given: $g(x) = -x^2$ where $x \leq 0$

(a) Write down the inverse of g, g^{-1} in the form $h(x) = ...$

(b) Sketch the graphs of g, h and $y = x$ on the same set of axis.

Solutions

(a) $y = -x^2$

$x = -y^2$

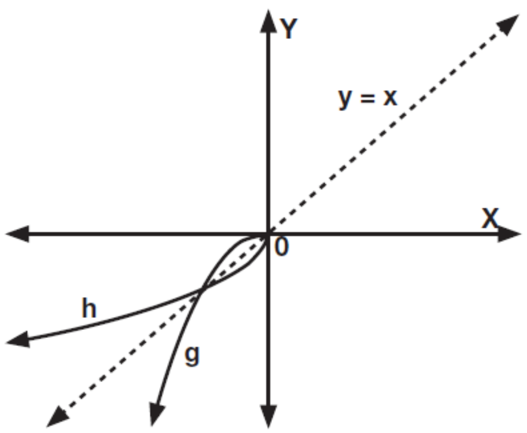
$-x = y^2$

$\pm\sqrt{-x} = y$

$-\sqrt{-x} = y$ where x ≤ 0

$\therefore h(x) = -\sqrt{-x}$

(b)



Inverse function : Exponential

$$y = b^x; (b > 0, b \neq 1)$$

Example 4

Given: $f(x) = 2^x$

a) Determine f^{-1} in the form $y = \ldots\ldots$

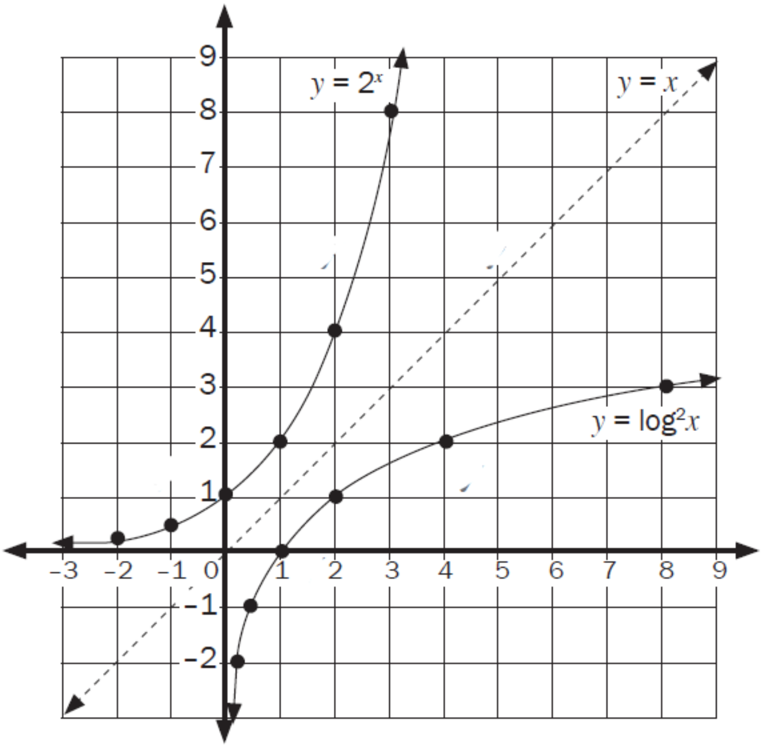
b) Sketch the graphs of $f(x)$, $f^{-1}(x)$ and $y = x$ on the same set of axes.

c) Write the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) The inverse of the exponential function $y = 2^x$ is $x = 2^y$ which can be written as $y = \log_2 x$.

b)



c) The domain and Range of $f(x)$

Domain: $x \in R$
Range: $y > 0$

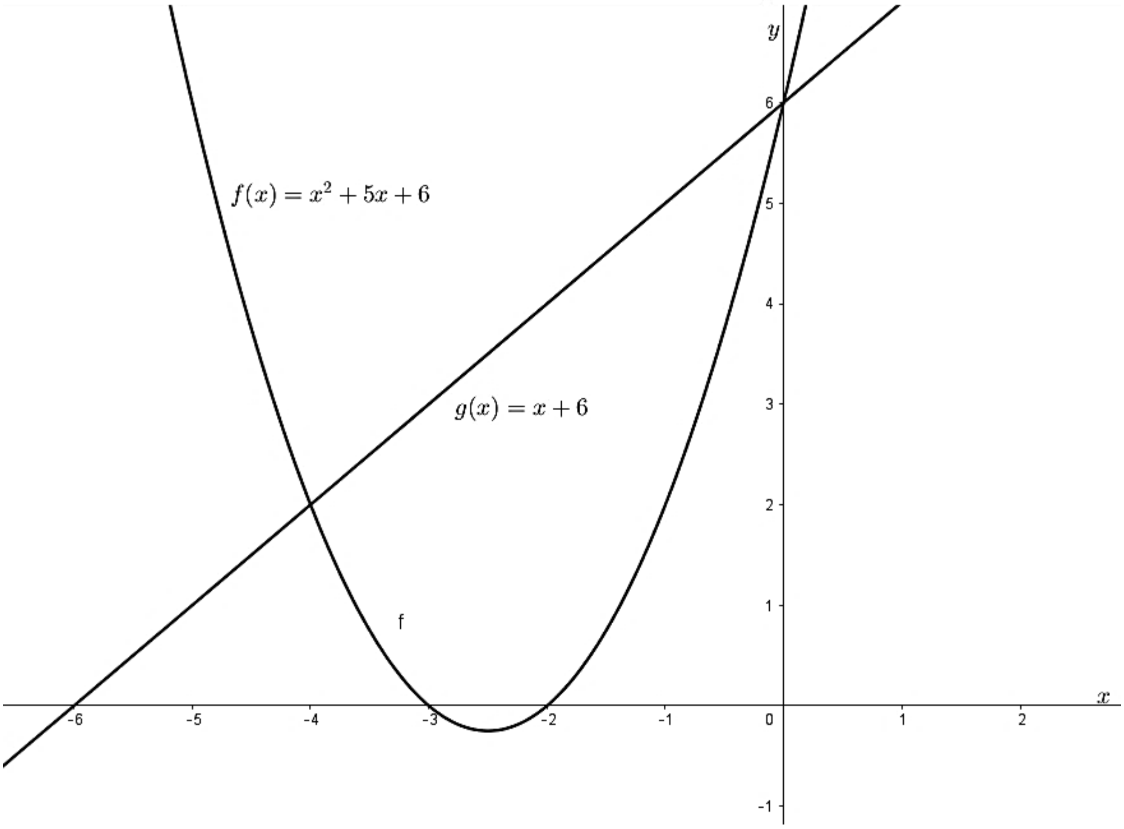
The domain and Range of $f^{-1}(x)$

Domain: $x > 0$
Range: $y \in 0$

SECTION 6: COMBINATIONS

Take note of the following when working with combination of functions:

Consider the graphs of f and g below

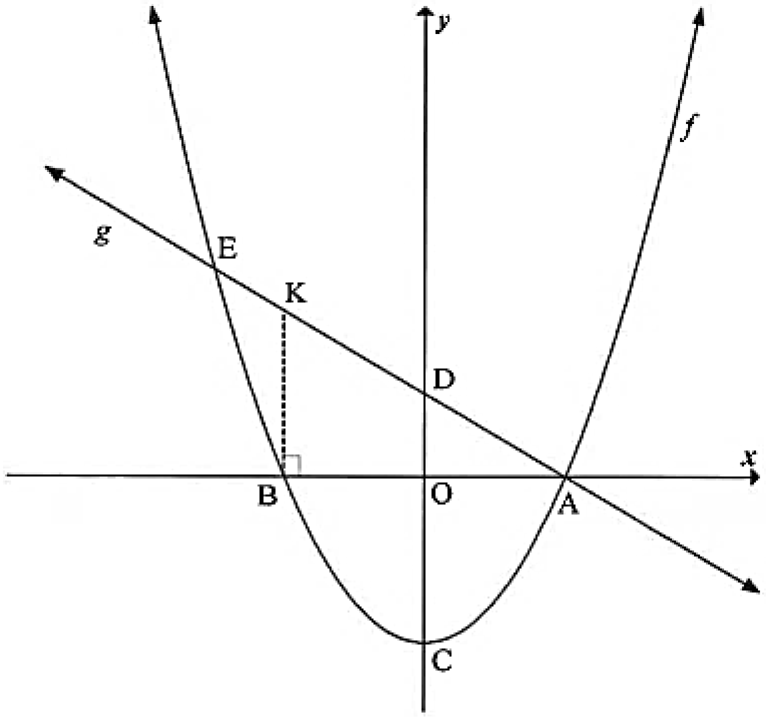


For $f(x) > 0$ *or* $f(x) < 0$	For $f(x) > g(x)$ *or* $f(x) < g(x)$	For $f(x).g(x) > 0$ *or* $f(x).g(x) < 0$
Focus on the $x - axis$. $f(x) > 0$ means where the graph of $f(x)$ is positive, which will be above the $x - axis$. And $f(x) < 0$ means where the graph of $f(x)$ is negative, which will be below the $x - axis$.	$f(x) > g(x)$ means where the graph of $f(x)$ is above the graph of $g(x)$. And $f(x) < g(x)$ means where the graph of $f(x)$ is below the graph of $g(x)$.	**$f(x).g(x) > 0$ means where the product of $f(x)$ and $g(x)$ is positive.** And **$f(x).g(x) < 0$ means where the product of $f(x)$ and $g(x)$ is negative.**

Example 1

QUESTION 5

The graphs of $f(x)=x^2-4$ and $g(x)=-x+2$ are sketched below. A and B are the x-intercepts of f. C and D are the y-intercepts of f and g respectively. K is a point on g such that BK $\parallel$ x-axis. f and g intersect at A and E.



5.1 Write down the coordinates of C.

5.2 Write down the coordinates of D.

5.3 Determine the length of CD.

5.4 Calculate the coordinates of B.

5.5 Determine the coordinates of E, a point of intersection of f and g.

5.6 For which values of x will:

5.6.1 $f(x) < g(x)$

5.6.2 $f(x).g(x) \geq 0$

5.7 Calculate the length of AK.

Solutions

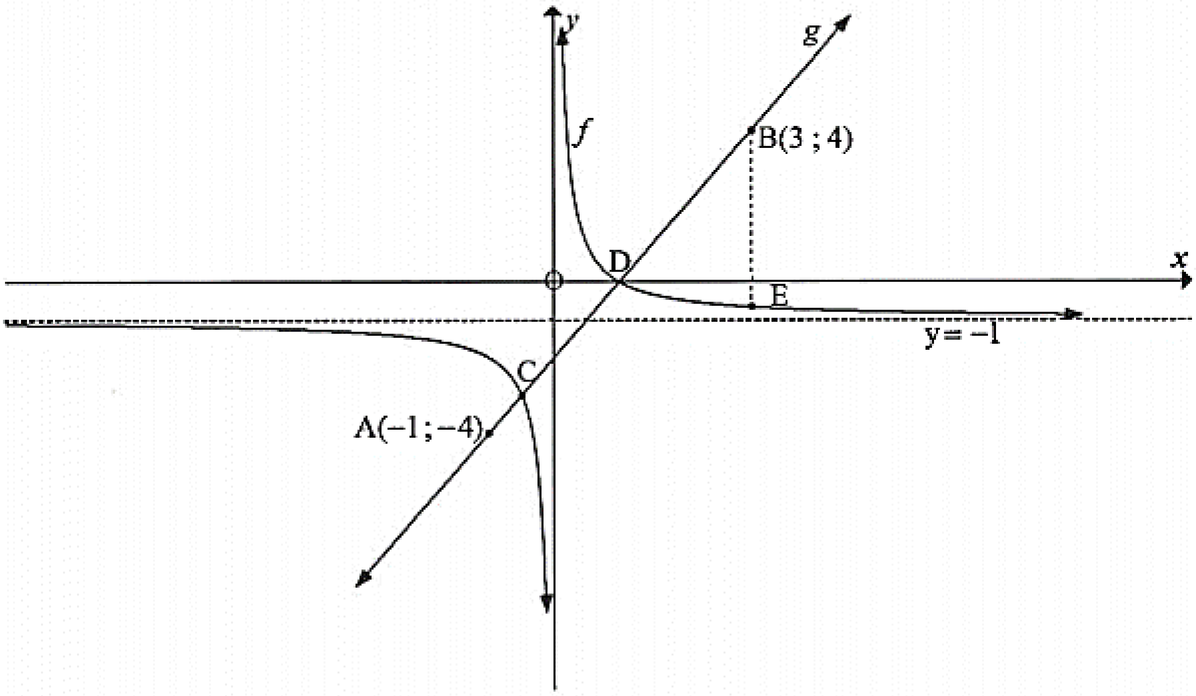
QUESTION 5	
5.1	C(0 ; –4)
5.2	D(0 ; 2)
5.3	CD = 2 – (–4) CD = 6 units/*eenhede*
5.4	$x^2 - 4 = 0$ $(x-2)(x+2) = 0$ $x = 2 \quad x = -2$ B(–2 ; 0)
5.5	$x^2 - 4 = -x + 2$ $x^2 + x - 6 = 0$ $(x-2)(x+3) = 0$ $x = 2 \quad x = -3$ E(–3 ; 5)
5.6.1	$-3 < x < 2$ **OR/*OF*** $(-3\ ;\ 2)$
5.6.2	$(-\infty\ ;\ -2] \cup \{2\}$
5.7	K(–2 ; 4) BK = 4 units/*eenhede* AB = 4 units/*eenhede* AK $= \sqrt{4^2 + 4^2}$ (Pythagoras) = 5,66 units/*eenhede*

Example 2

QUESTION 5

The sketch below shows f and g, the graphs of $f(x)=\frac{1}{x}-1$ and $g(x)=ax+q$ respectively.

Points A$(-1\,;-4)$ and B$(3\,;4)$ lie on the graph g.
The two graphs intersect at points C and D.
Line BE is drawn parallel to the y-axis, with E on f.



5.1 Show that $a=2$ and $q=-2$.

5.2 Determine the values of x for which $f(x)=g(x)$.

5.3 For what values of x is $g(x)\geq f(x)$?

5.4 Calculate the length of BE.

5.5 Write down an equation of h if $h(x)=f(x)+3$.

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Solutions

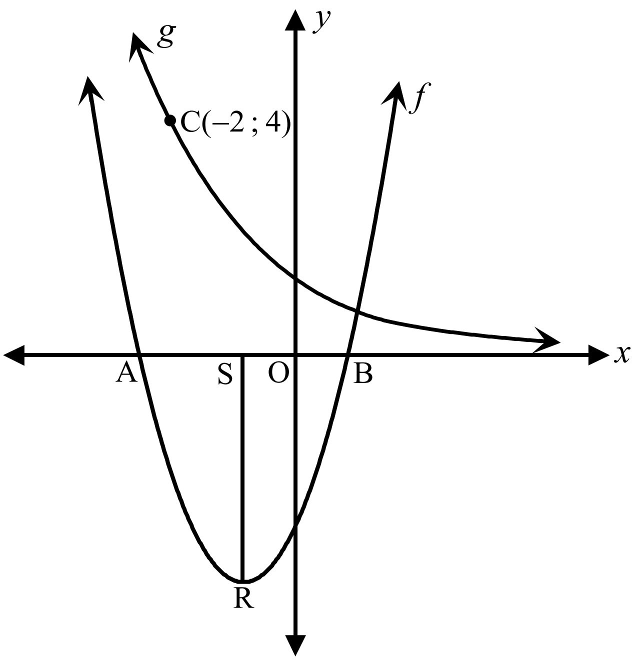
5.1	$a = \text{gradient of } g$ $= \dfrac{-4-4}{-1-3}$ $= 2$ $4 = 2(3) + q$ $q = -2$ $g(x) = 2x - 2$
5.2	$\dfrac{1}{x} - 1 = 2x - 2$ $\dfrac{1}{x} = 2x - 1$ $1 = 2x^2 - x$ $2x^2 - x - 1 = 0$ $(2x+1)(x-1) = 0$ $x = -\dfrac{1}{2}$ or $x = 1$

5.3	$-\frac{1}{2} \leq x < 0$ or/*of* $x \geq 1$ **OR/OF** $\left[-\frac{1}{2}\,;\,0\right) \cup [1\,;\,\infty)$
5.4	$f(3) = \frac{1}{3} - 1$ $= -\frac{2}{3}$ Length of BE $= 4 - f(3)$ $= 4 - \left(-\frac{2}{3}\right)$ $= 4 + \frac{2}{3}$ $= 4\frac{2}{3}$ **OR/OF** $\text{BE} = 2x - 2 - \frac{1}{x} + 1$ $= \frac{2x^2 - x - 1}{x}$ $(x = 3)\ \text{BE} = \frac{2(3)^2 - (3) - 1}{3}$ $= \frac{18 - 4}{3}$ $= 4\frac{2}{3}$
5.5	$h(x) = f(x) + 3$ $h(x) = \frac{1}{x} + 2$

Example 3

QUESTION 1

The graphs of $f(x) = 2x^2 + 4x - 6$ and $g(x) = a^x$ are represented in the sketch below. A and B are the x-intercepts of f and R is the turning point of f. The point $C(-2\ ;\ 4)$ is a point on the graph of g.



1.1 Show that $a = \frac{1}{2}$

1.2 Determine the length of AB.

1.3 Determine the length of SR.

1.4 Write down the equation of h, if h is the reflection of f in the y-axis. Express your answer in the form $a = \frac{1}{2}h(x) = a(x + p)^2 + q$.

1.5 Write down the equation of g^{-1} in the form $y = \ldots$.

1.6 Sketch the graph of $y = g^{-1}(x)$ on a set of axes.

1.7 Determine the values of x for which:

1.7.1 $g^{-1}(x) \geq -2$

Solutions

1.1 $y = a^x$

$\therefore 4 = a^{-2}$

$\therefore 4 = \dfrac{1}{a^2}$

$\therefore 4a^2 = 1$

$\therefore a^2 = \dfrac{1}{4}$

$\therefore a = \dfrac{1}{2}$

1.2 $0 = 2x^2 + 4x - 6$

$\therefore 0 = x^2 + 2x - 3$

$\therefore 0 = (x+3)(x-1)$

$\therefore x = -3$ or $x = 1$

$\therefore$ AB = 4 units

1.3 $x_R = -\dfrac{4}{2(2)} = -1$

$\therefore y_R = 2(-1)^2 + 4(-1) - 6 = -8$

$\therefore$ SR = 8 units

Alternatively:

$f'(x) = 4x + 4$

$\therefore 0 = 4x + 4$

$\therefore x = -1$

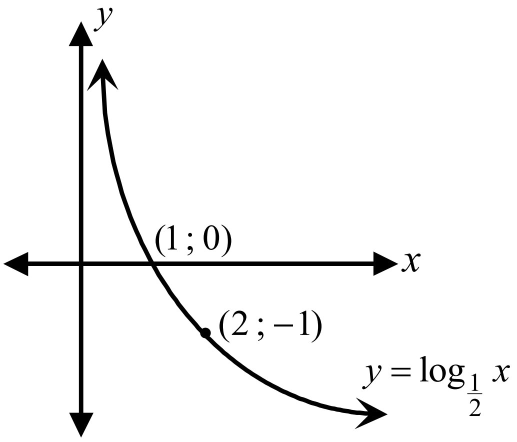
1.4 $h(x) = 2(x-1)^2 - 8$

1.5 $y = \left(\dfrac{1}{2}\right)^x$

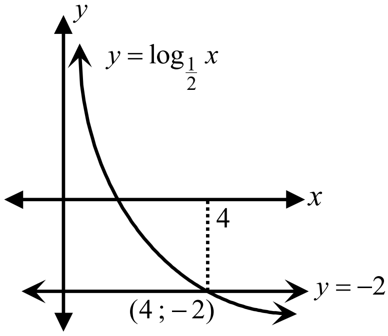
$\therefore x = \left(\dfrac{1}{2}\right)^y$

$\therefore y = log_{\frac{1}{2}} x$

1.6



1.7.1



$$\log_{\frac{1}{2}} x = -2$$

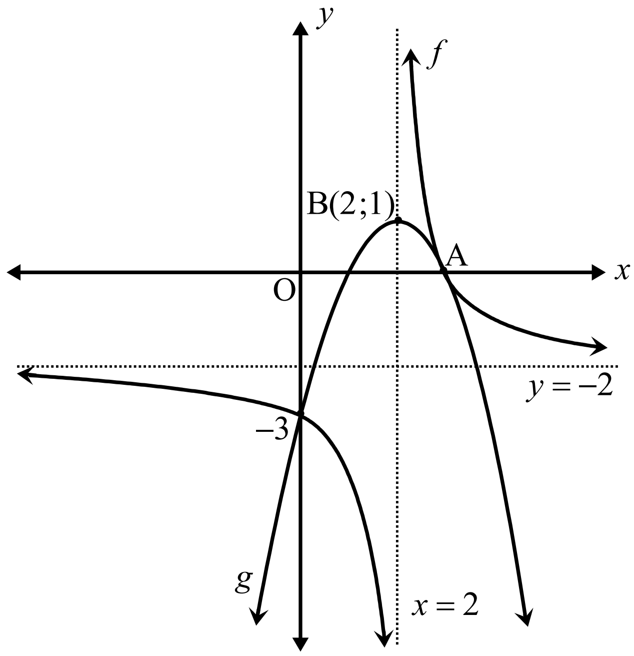
$$\therefore x = \left(\frac{1}{2}\right)^{-2} = 4$$

$$\therefore g^{-1}(x) \geq -2 \text{ for all } 0 < x \leq 4$$

Example 4

QUESTION 1

In the diagram below, the graph of $f(x) = \frac{a}{x+p} + q$ cuts the y-axis at -3 and the x-axis at A. The graph of $g(x) = m(x+n)^2 + c$ intersects f at A, cuts the y-axis at -3 and has a turning point at $B(2; 1)$.



Determine:

1.1 the equation of *f*.

1.2 the equation of *g*.

1.3 the length of OA.

1.4 the values of x for which $f(x).g(x) \leq 0$.

Solutions

1.1

$$y = \frac{a}{x-2} - 2$$

Substitute $(0\ ;\ -3)$:

$$-3 = \frac{a}{0-2} - 2$$

$$\therefore -1 = \frac{a}{-2}$$

$$\therefore a = 2$$

$$\therefore f(x) = \frac{2}{x-2} - 2$$

1.2 $y = m(x-2)^2 + 1$

Substitute $(0\ ;\ -3)$:

$\therefore -3 = m(0-2)^2 + 1$

$\therefore -4 = m(-2)^2$

$\therefore -4 = 4m$

$\therefore m = -1$

$\therefore g(x) = -(x-2)^2 + 1$

1.3

$$0 = \frac{2}{x-2} - 2$$

$\therefore 0 = 2 - 2(x-2)$

$\therefore 0 = 2 - 2x + 4$

$\therefore 2x = 6$

$\therefore x = 3$

$\therefore \text{OA} = 3$ units

Alternatively:

$0 = -(x-2)^2 + 1$

$\therefore (x-2)^2 = 1$

$\therefore x^2 - 4x + 4 = 1$

$\therefore x^2 - 4x + 3 = 0$

$\therefore (x-1)(x-3) = 0$

$x = 1$ or $x = 3$

$\therefore \text{OA} = 3$ units

1.4 $f(x).g(x) \leq 0$ for all

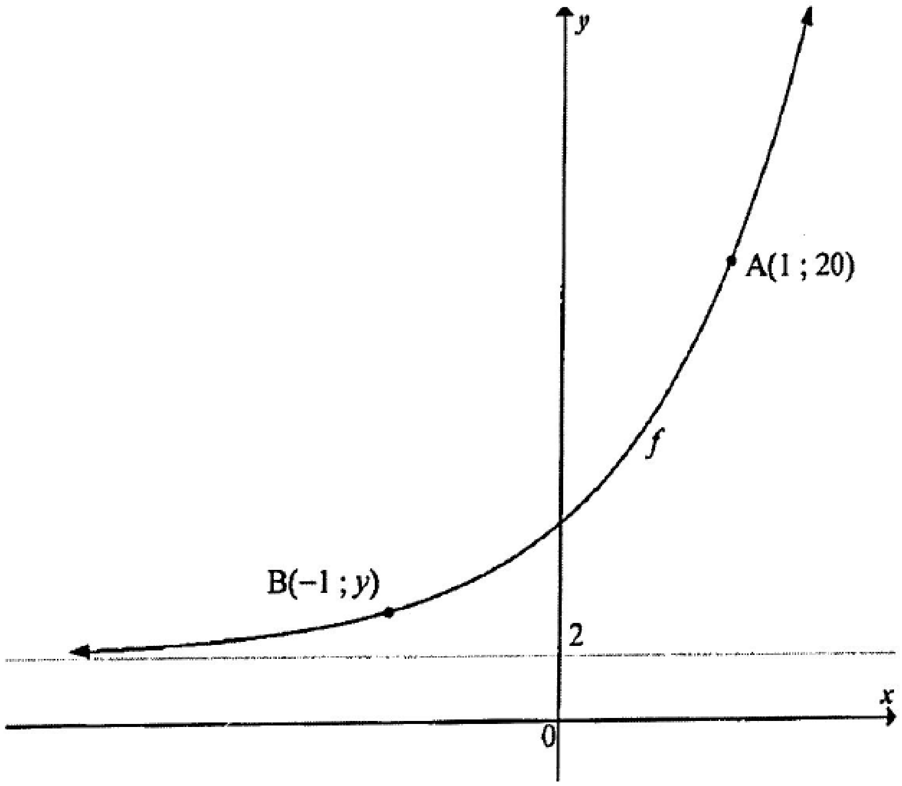
$1 \leq x < 2$ or $x = 3$

SECTION 7: THE AVERAGE GRADIENT BETWEEN TWO POINTS

The **average gradient** of a function between any two points is defined to be the **gradient of the line** joining the two points.

Example

The sketch below is the graph of $f(x) = 2.b^{x+1} + q$.
The graph of f passes through the points A(1 ; 20) and B (–1 ; y).
The line $y = 2$ is an asymptote of f.



1. Show that the equation of f is $f(x) = 2(3)^{x+1} + 2$

2. Calculate the y-coordinate of the point B.

3. Determine the average gradient of the curve between the points A and B.

Solutions

1.
$$q = 2$$
$$f(x) = 2.b^{x+1} + 2$$
$$20 = 2.b^{1+1} + 2$$
$$18 = 2.b^2$$
$$9 = b^2$$
$$b = 3$$
$$f(x) = 2.3^{x+1} + 2$$

2.
$$y = 2.3^{-1+1} + 2$$
$$y = 2.1 + 2$$
$$y = 4$$

3.
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{20-4}{1-(-1)}$$
$$= 8$$

SECTION 8: ACTIVITIES OF FUNCTIONS AND GRAPHS

Activity 1

QUESTION 4

4.1 Given the function $p(x)=\left(\frac{1}{3}\right)^x$.

4.1.1 Is p an increasing or decreasing function? (1)

4.1.2 Determine p^{-1}, the inverse of p, in the form $y=\ldots$ (2)

4.1.3 Write down the domain of p^{-1}. (1)

4.1.4 Write down the equation of the asymptote of $p(x)-5$. (1)

4.2 Given: $f(x)=\frac{4}{x-1}+2$

4.2.1 Write down the equations of the asymptotes of f. (2)

4.2.2 Calculate the x-intercept of f. (2)

4.2.3 Sketch the graph of f, label all asymptotes and indicate the intercepts with the axes. (4)

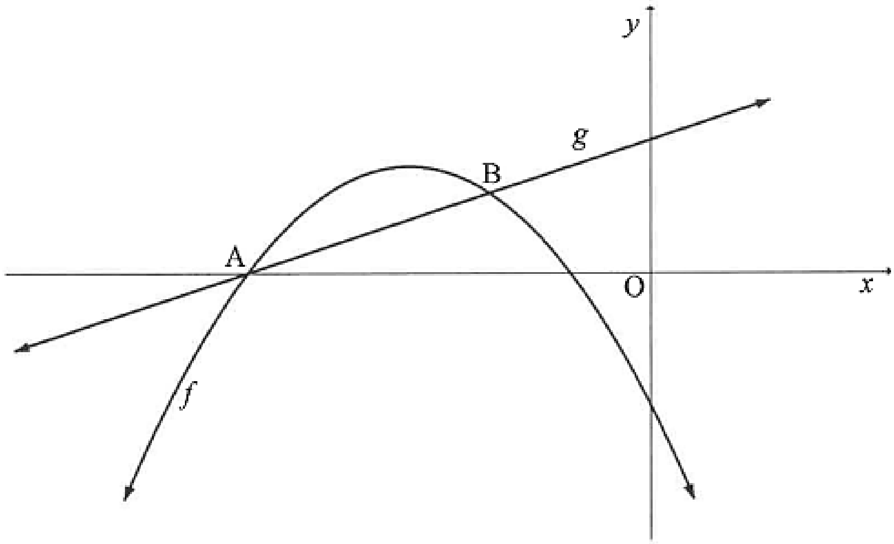
4.2.4 Use your graph to determine the values of x for which $\frac{4}{x-1}\geq -2$. (2)

4.2.5 Determine the equation of the axis of symmetry of $f(x-2)$, that has a negative gradient. (3)

[18]

QUESTION 5

The graphs of the functions $f(x) = -(x+3)^2 + 4$ and $g(x) = x + 5$ are drawn below.
The graphs intersect at A and B.



5.1	Write down the coordinates of the turning point of f.	(2)
5.2	Write down the range of f.	(1)
5.3	Show that the x-coordinates of A and B are -5 and -2 respectively.	(4)
5.4	Hence, determine the values of c for which the equation $-(x+c+3)^2+4=(x+c)+5$ has ONE negative and ONE positive root.	(2)
5.5	The maximum distance between f and g in the interval $x_A < x < x_B$ is k. If $h(x) = g(x) + k$, determine the equation of h in the form $h(x) = ...$	(5)

[14]

Activity 2

QUESTION 4

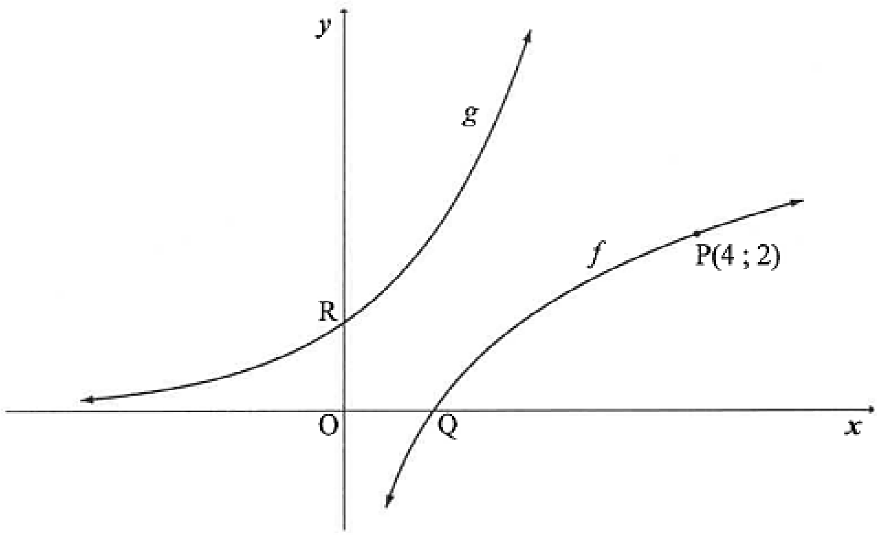
Given: $g(x) = \dfrac{1}{x-1} + 2$

4.1	Write down the equations of the asymptotes of g.	(2)
4.2	Draw a graph of g, indicating any intercepts with the axes and asymptotes.	(4)
4.3	Determine the values of x where $g(x) > 0$.	(2)
4.4	Determine the equation of the axis of symmetry of g which has a negative gradient.	(2)

[10]

QUESTION 5

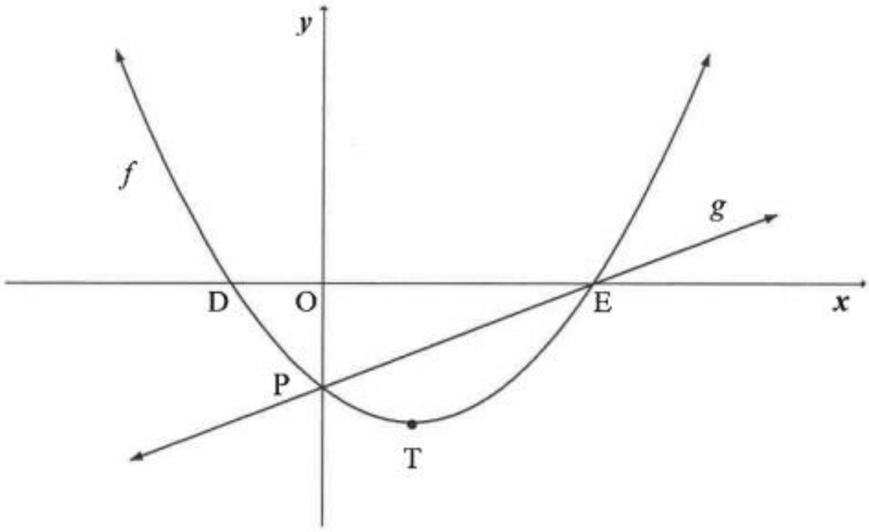
In the diagram, the graphs of $f(x)=\log_a x$ and g are drawn. Graph g is the reflection of f in the line $y=x$. Graph f passes through the point P(4 ; 2). Q is the x-intercept of f and R is the y-intercept of g.



5.1	Write down the coordinates of P$'$, the image of P on g.	(2)
5.2	Show that $a=2$.	(2)
5.3	Write down the equation of g in the form $y=\ldots$	(1)
5.4	T is a point on f in the first quadrant where TR is parallel to the x-axis. Calculate the area of ΔRTP$'$.	(4)
		[9]

QUESTION 6

The graphs of $f(x) = x^2 - 2x - 3$ and $g(x) = mx + c$ are drawn below. D and E are the x-intercepts and P is the y-intercept of f. The turning point of f is T(1 ; – 4). The graphs of f and g intersect at P and E.



6.1 Write down the range of f. (1)

6.2 Calculate the coordinates of D and E. (3)

6.3 Determine the equation of g. (2)

6.4 Write down the values of x for which $f(x) - g(x) > 0$. (2)

6.5 Determine the maximum vertical distance between h and g if $h(x) = -f(x)$ for $x \in [-2 ; 3]$. (5)

6.6 Given: $k(x) = g(x) - n$.

Determine n if k is a tangent to f. (5)

[18]

Activity 3

QUESTION 4

Given: $f(x) = a^x - 1$ for $a > 0$. $B\left(2 ; \frac{-5}{9}\right)$ is a point on f.

4.1 Calculate the value of a. (2)

4.2 Write down the range of f. (1)

4.3 Sketch the graph of f. Clearly show the intercepts with the axes and asymptotes, if any. (3)

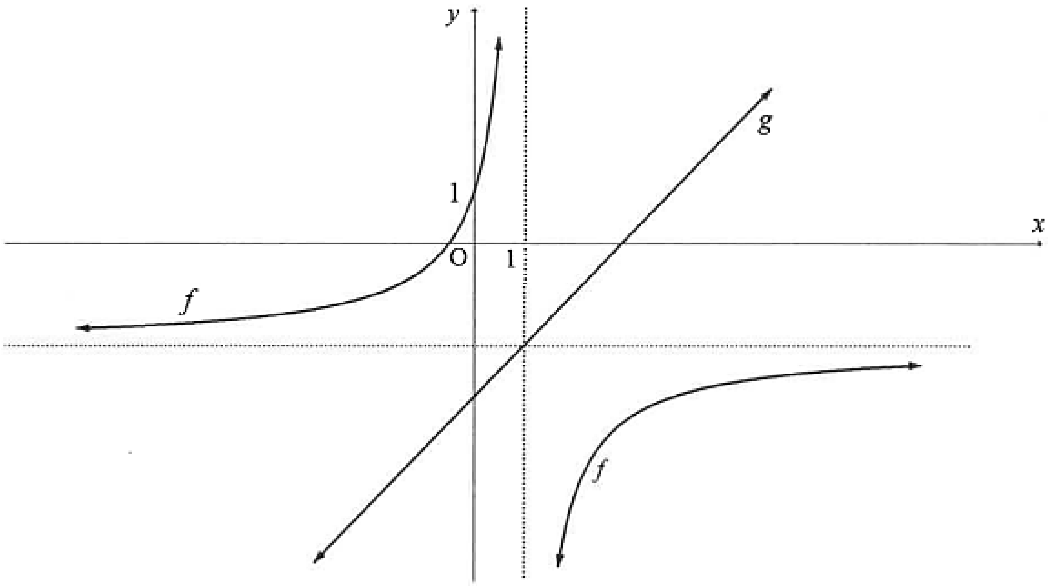
4.4 It is further given that C is a point on f at $y = \frac{19}{8}$.

Determine the coordinates of C', the image of C, when C is reflected about the line $y = x$. (3)

[9]

QUESTION 5

Sketched below is the graph of $f(x) = \frac{a}{x+p} + q$ having the domain $(-\infty ; 1) \cup (1 ; \infty)$.
The graph of f cuts the y-axis at $(0 ; 1)$. A line of symmetry of f is given by $g(x) = x - 3$.



5.1 Write down the value of p. (1)

5.2 Determine the equation of the horizontal asymptote of f. (2)

5.3 Calculate the value of a. (2)

5.4 For which values of x is $f(x) \geq 0$? (3)

5.5 Graph f undergoes a transformation to h where:

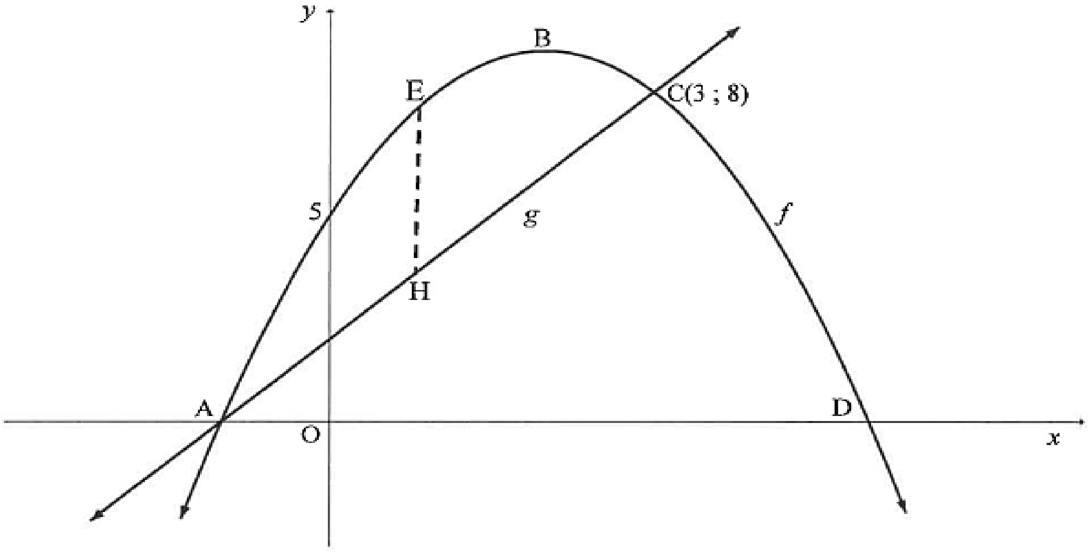
- The domain and range of h are the same as that of f
- $h'(x)$, the derivative of h, is negative on its domain

Describe a possible transformation that f could have undergone to result in h. (2)

[10]

QUESTION 6

In the diagram below, the graphs of $f(x) = -x^2 + 4x + 5$ and g, a straight line, are drawn. $C(3\,;8)$ is a point of intersection of f and g. EH is drawn parallel to the y-axis, with E a point on f and H a point on g.



6.1 Calculate the coordinates of B, the turning point of f. (3)

6.2 Show that the equation of the line through A and C is given by $g(x) = 2x + 2$. (3)

6.3 Calculate the maximum length of EH for $f > g$. (4)

6.4 Given: $k(x) = f(x+m) = -x^2 - 2mx - m^2 + 4x + 4m + 5$

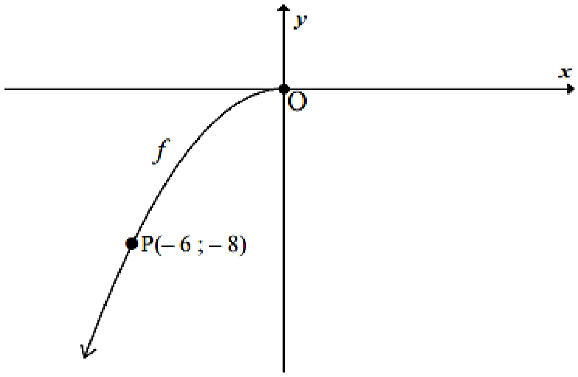
Determine the value of m such that g is a tangent to k. (5)

[15]

Activity 4

QUESTION 6

The graph of $f(x) = ax^2$, $x \leq 0$ is sketched below.
The point P(– 6 ; – 8) lies on the graph of *f*.



6.1 Calculate the value of a. (2)

6.2 Determine the equation of f^{-1}, in the form $y = \ldots$ (3)

6.3 Write down the range of f^{-1}. (1)

6.4 Draw the graph of f^{-1} **ON YOUR BOOK** . Indicate the coordinates of a point on the graph different from (0 ; 0). (2)

6.5 The graph of f is reflected across the line $y = x$ and thereafter it is reflected across the x-axis. Determine the equation of the new function in the form $y = \ldots$ (3)

[11]

ANNEXURE A: INFORMATION SHEET

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$A=P(1+ni)$ $\quad$ $A=P(1-ni)$ $\quad$ $A=P(1-i)^n$ $\quad$ $A=P(1+i)^n$

$T_n=a+(n-1)d$ $\quad$ $S_n=\frac{n}{2}[2a+(n-1)d]$

$T_n=ar^{n-1}$ $\quad$ $S_n=\frac{a(r^n-1)}{r-1}$; $\quad r\neq 1$ $\quad$ $S_\infty=\frac{a}{1-r}$; $-1<r<1$

$F=\frac{x[(1+i)^n-1]}{i}$ $\quad$ $P=\frac{x[1-(1+i)^{-n}]}{i}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ $\quad$ $\mathrm{M}\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$

$y=mx+c$ $\quad$ $y-y_1=m(x-x_1)$ $\quad$ $m=\frac{y_2-y_1}{x_2-x_1}$ $\quad$ $m=\tan\theta$

$(x-a)^2+(y-b)^2=r^2$

$In\ \Delta ABC:\ \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ $\quad$ $a^2=b^2+c^2-2bc.\cos A$ $\quad$ $area\ \Delta ABC=\frac{1}{2}ab.\sin C$

$\sin(\alpha+\beta)=\sin\alpha.\cos\beta+\cos\alpha.\sin\beta$ $\quad$ $\sin(\alpha-\beta)=\sin\alpha.\cos\beta-\cos\alpha.\sin\beta$

$\cos(\alpha+\beta)=\cos\alpha.\cos\beta-\sin\alpha.\sin\beta$ $\quad$ $\cos(\alpha-\beta)=\cos\alpha.\cos\beta+\sin\alpha.\sin\beta$

$$\cos 2\alpha=\begin{cases}\cos^2\alpha-\sin^2\alpha\\1-2\sin^2\alpha\\2\cos^2\alpha-1\end{cases}$$

$\sin 2\alpha=2\sin\alpha.\cos\alpha$

$\bar{x}=\frac{\sum fx}{n}$ $\quad$ $\sigma^2=\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}$

$P(A)=\frac{n(A)}{n(S)}$ $\quad$ $P(A\text{ or }B)=P(A)+P(B)-P(A\text{ and }B)$

$\hat{y}=a+bx$ $\quad$ $b=\frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$

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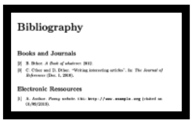
ANNEXURE B

3. ELABORATION OF CONTENT/TOPICS

The purpose of the clarification of the topics is to give guidance to the teacher in terms of depth of content necessary for examination purposes. Integration of topics is encouraged as learners should understand Mathematics as a holistic discipline. Thus questions integrating various topics can be asked.

FUNCTIONS

1. Candidates must be able to use and interpret functional notation. In the teaching process learners must be able to understand how $f(x)$ has been transformed to generate $f(-x)$, $-f(x)$, $f(x+a)$ $f(x)+a$, $af(x)$ and $x = f(y)$ where $a \in R$.
2. Trigonometric functions will ONLY be examined in PAPER 2.



BIBLIOGRAPHY

1	**CAPS Document**
2	**Examination guidelines**
3	**Dbe November 2008 – 2024 Past Papers**
4	**Dbe June and Feb/Mar 2009 – 2024 Past Papers**
5	**Mind The Gap Textbook**
6	**Mind Action Series Textbook**
7	**JENN 2022 – 2024 Manuals**

JENN

Training and Consultancy

The path to enlightened education

SUBJECT: MATHEMATICS

GRADE 12

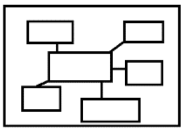
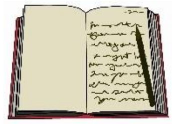
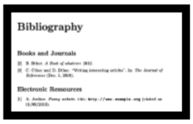
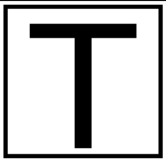
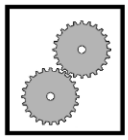
TERM 1

TEACHER AND LEARNER CONTENT MANUAL

TOPIC

TRIGONOMETRY

ICON DESCRIPTION

 MIND MAP	 EXAMINATION GUIDELINE	 CONTENTS	 ACTIVITIES
 BIBLIOGRAPHY	 TERMINOLOGY	 WORKED EXAMPLES	 STEPS

CONTENTS PAGE

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SECTION 1: POLICY

Policy Guidelines(outcomes): TRIGONOMETRY

A	B	C
(a) Definitions of the trigonometric ratios sin θ, cos θ and tan θ in a right-angled triangles. (b) Extend the definitions of sin θ, cos θ and tan θ to $0° \leq \theta \leq 360°$ (c) Derive and use values of the trigonometric ratios (without using a calculator for the special angles $\theta \in \{0°;30°;45°;60°;90°\}$. (d) Define the reciprocals of trigonometric ratios.	(a) derive and use the identities: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \sin^2\theta = 1$. (b) derive the reduction formulae (c) determine the general solution and / or specific solutions of trigonometric equations (d) establish the sine, cosine and area rules	Proof and use of the compound angle and double angle identities. Solve problems in two and three dimensions applying the sine, cosine and area rules.
Solve problems in two dimensions.	Solve problems in 2-dimensions.	

(SOURCE: CURRICULUM AND ASSESSMENT POLICY STATEMET(CAPS) SENIOR PHASE GRADE 10-12 MATHEMATICS)

Concepts and mark allocation:

Concept(s)	Grade	Marks
Trigonometry – solving triangles and Identities	10,11 & 12	± 12
Trigonometry - Reduction Formula and Equations	10,11 & 12	± 18
Trigonometry Graphs	10 &11	± 12
Trigonometry 2D and 3D	10,11 & 12	± 8
TOTAL		**50**

Diagnostic report recommendations:

- When solving right angled triangles, use Pythagoras theorem to find all sides to apply trigonometric ratios; x, y and r.
- Teaching must explain the reduction of co-functions in relation to the quadrants.
- General solutions must be explained using the reference angle to find the value of the required angles. Explain when and why is the solution general and when not.
- Teaching must consider analysis and interpretation of graphs.
- Identify triangles with a diagram(text) and classify as follows:
  - ✓ Right angled- triangles – Trig ratios and Pythagoras
  - ✓ area rule – when the word area is mentioned
  - ✓ cosine rule – given three sides of a triangle
  - ✓ sine rule – any condition that does not satisfy the cosine rule.

T

	Word	Description / Definition	Symbols / Formula
1	Reduction Formulae	Are used to reduce any angle greater than 90^0 to an acute angle	$90^0 \pm \theta$ $180^0 \pm \theta$ $360^0 \pm \theta$
2	Co-function	Two angles are said to be complementary angles if their sum is equal to 90°.	$\sin\theta = \cos(90° - \theta)$ $\cos\theta = \sin(90° - \theta)$ $\sin(90° + \theta) = \cos\theta$ $\cos(90° + \theta) = -\sin\theta$
3	Trig-identity	Is a statement of equality with a left-hand side and can be proved by using certain identities which are derived from the basic trig-functions	$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\sin^2\theta + \cos^2\theta = 1$ $\sin^2\theta = 1 - \cos^2\theta$ $\cos^2\theta = 1 - \sin^2\theta$
4	Restriction	An identity is true for all values of the variable except for those values the identity is not define for or undefined.	
5	General solution	To find all angles (also larger than 360^0 and smaller than 0^0) that will satisfy the equation	If it is a sine or cosine function, add $k.360°, k \in Z$and if it is a tangent function, add$k.180°, k \in Z$
6	Compound angles	The trigonometric identities $\cos(A - B) = \text{cosAcosB} + \text{sinAsinB}$ $\cos(\text{A+B}) = \text{cosAcosB} - \text{sinAsinB}$ $\sin(A - B) = \text{sinAcosB} - \text{sinBcosA}$ $\sin(A + B) = \text{sinAcosB} + \text{sinBcosA}$	
7	Double angles	Trigonometric ratio of $2x$ an angle	$\text{cos2A} = \cos^2 A - \sin^2 A$ $\text{cos2A} = 1 - 2\sin^2 A$ $\text{cos2A} = 2\cos^2 A - 1$ $\text{sin2A} = \text{2sinAcosA}$
8	Period	If a sine, cosine or tangent graph completes a full cycle, in a certain interval, measured in degrees .	sine and cosine graphs: $\frac{360^0}{b}$

9	Amplitude	It is $\frac{1}{2}$ distance between the maximum and minimum y-value of the graph and is always a positive value	
10	Solve a triangle	If a triangle is not right-angled, any sides or angles can be determined by the sine, cosine or area rules. The triangle can be solved if 3 values are given. $\text{sine}-\text{rule:}\quad \text{s s}\angle\,;\angle\angle s$ $\text{cosine}-\text{rule:}\ \text{s}\angle\text{s}\,;\text{s s s}$ $\text{area}-\text{rule:}\ \text{s}\angle s$	Sine rule: $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$ Cosine rule: b^2 $=a^2+c^2-2.\text{a.c.CosA}$ Area rule: $\text{Area }\Delta\text{ABC}=\frac{1}{2}\text{ab SinC}$

SECTION 2: Trigonometry Ratios, Cartesian plane and Identities

Trigonometry Ratios and Identities

Content and worked examples:

There are three special ratios in the study of Trigonometry, namely the sine, cosine and tangent ratios. These ratios are applied to a right-angled triangle, they define the relationships between the sides and angles.

We can define the sine, cosine and tangent of an angle as follows:

1.

$$\text{sine of an angle } \theta = \frac{\text{length of the side opposite angle } \theta}{\text{length of the hypotenuse}}$$

$$\therefore \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

2.

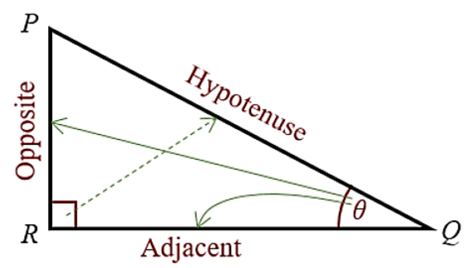
$$\text{cosine of an angle } \theta = \frac{\text{length of the side adjacent to angle } \theta}{\text{length of the hypotenuse}}$$

$$\therefore \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

3

$$\text{tangent of an angle } \theta = \frac{\text{length of the side opposite angle } \theta}{\text{length of the side adjacent to angle } \theta}$$

$$\therefore \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



Right-angled triangles

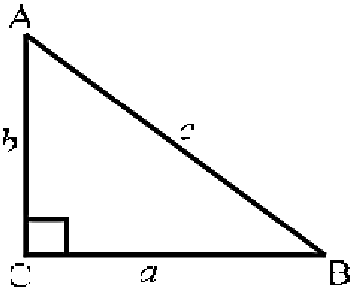
These triangles are fundamental to the study of trigonometry. In earlier grades you were introduced to the theorem of Pythagoras which describes the relationship between the three sides of a right-angled triangle.

From the Theorem of Pythagoras:

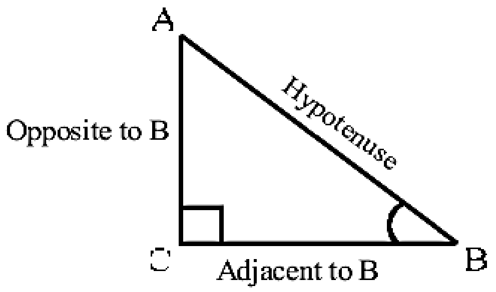
$$c^2 = a^2 + b^2$$

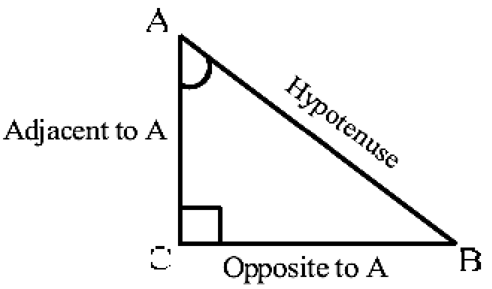
$$a^2 = c^2 - b^2$$

$$b^2 = c^2 - a^2$$



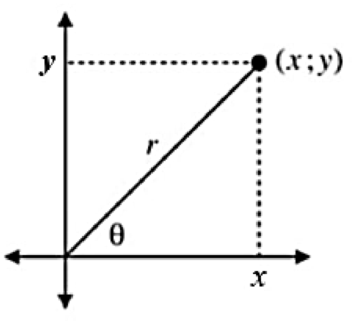
Furthermore, we will label the three sides as **adjacent, opposite and hypotenuse**. The longest side (opposite the right-angle) is called the hypotenuse. The opposite and adjacent sides are dependent on which angle is used as the reference point.





Cartesian Plane

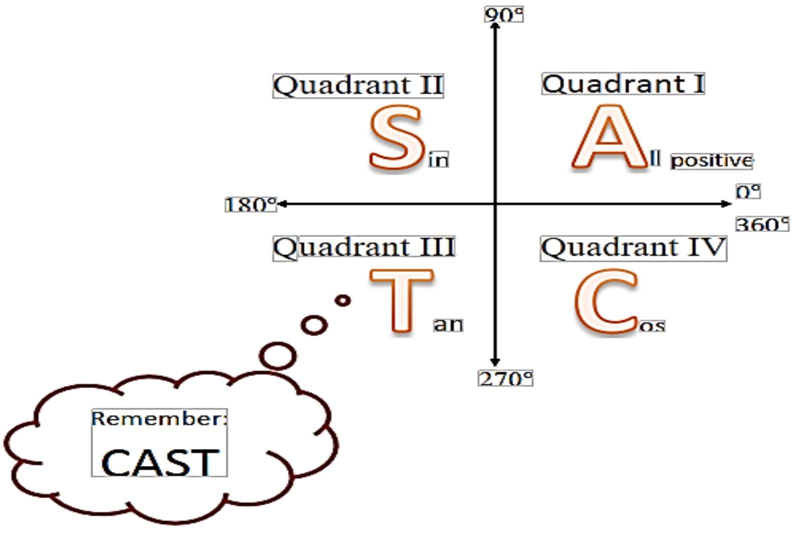
In a cartesian plane the Trigonometric ratios are defines as:



$\boxed{\sin\theta = \frac{y}{r}}$ $\boxed{\cos\theta = \frac{x}{r}}$ $\boxed{\tan\theta = \frac{y}{x}}$

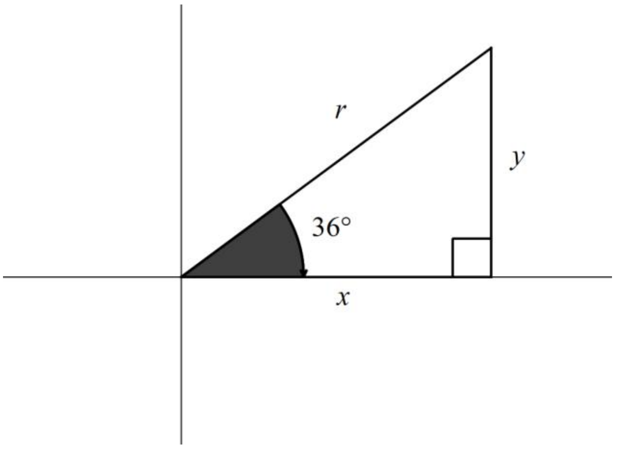
The angle formed will determine the sign of each trigonometric function in that quadrant.

- All trigonometric functions are positive in the first quadrant.
- sin θ is positive in the second quadrant and tan θ and cos θ are negative.
- tan θ is positive in the third quadrant and sin θ and cos θ are negative.
- cos θ is positive in the fourth quadrant and sin θ and tan θ are negative.

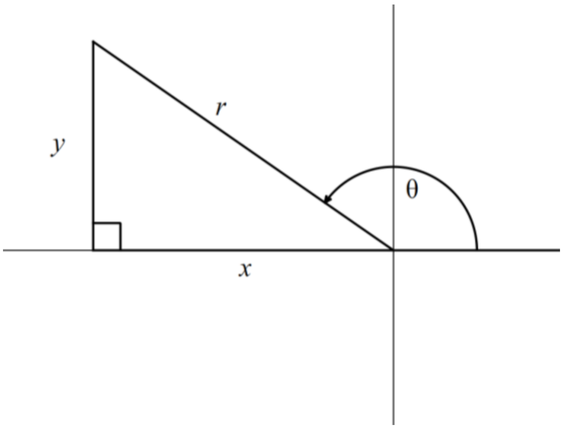


The following diagrams will show us how to draw our right-angled triangle in different quadrants.

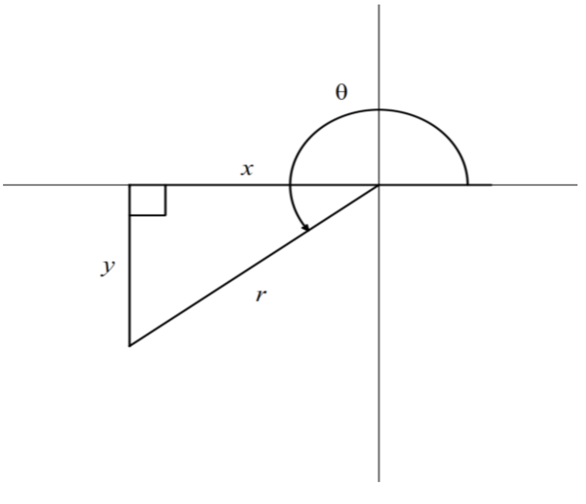
1st Quadrant



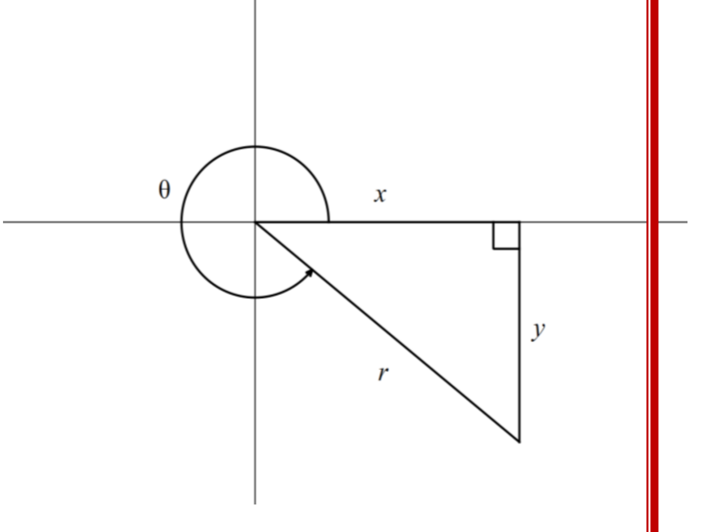
2nd Quadrant



3rd Quadrant



4th Quadrant



Worked example 1

Determine the quadrant which satisfies the following conditions and draw the accurate triangle.

a. $sin\ \theta < 0\ and\ \ 90° < \theta < 270°.$

b. $cos\theta > 0\ and\ tan\ \theta < 0$

c. $7\ tan\ \theta +\ 3 = 0\ and\ \theta\ \in\ [90°: 270°,]$

Solutions

a. $sin\ \theta\ <\ 0\ and\ 90° <\ \theta\ < 270°.$

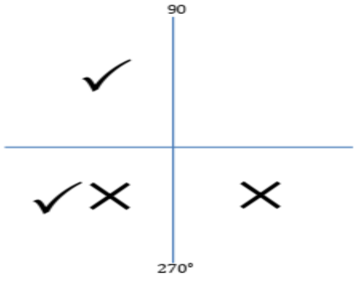
Step 1:

→use conditions to identify the quadrants and tick them with different symbols on the Cartesian plane.

Sin is negative in the 3rd and 4th quadrants.

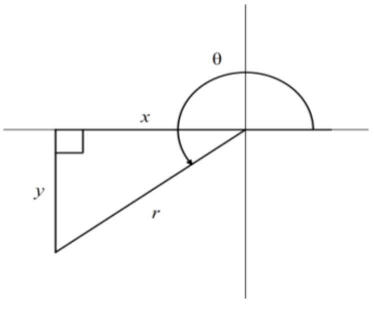
This interval represents 2nd and 3rd quadrants

Given that sin θ < 0° and 90° $<\ \theta\ < 270°.$



Where X represents the quadrant, we obtained from sin and tick is the quadrants obtained from the interval. We can notice the 3rd quadrant it satisfies both conditions, hence the terminal arm will be in the 3rd quadrant.

→ Complete the right-angled triangle.

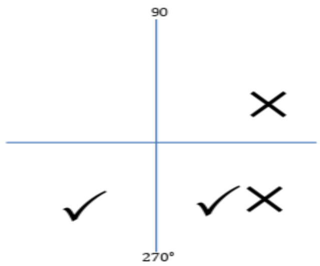


b. cos θ > 0 and tan θ < 0

cos is positive in the 1st and 4th quadrants.

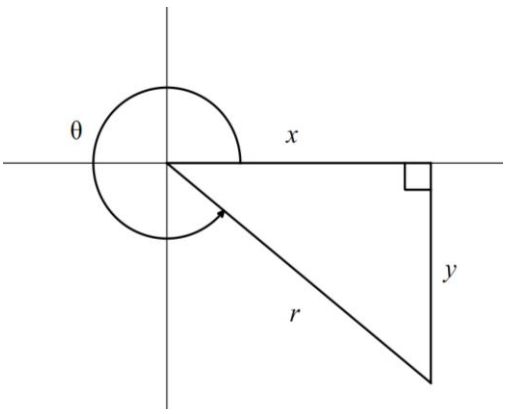
tan is negative in the 2nd and 4th quadrants

Given that cos θ > 0 and tan θ < 0.



Where X represents the quadrant, we obtained from cos and tick is the quadrants obtained from the tan. We can notice the 4th quadrant it satisfies both conditions, hence the terminal arm will be in the 4th quadrant.

→ Complete the right-angled triangle.



c. $7\ tan\ \theta + 3 = 0\ and\ \theta\ \in$ [90°: 270°,]

Answer:

The first step is to simplify the first equation in such a way that we only have trig ratio (tan θ) on the left-hand side.

$$7tan\ \theta\ + 3 = 0$$

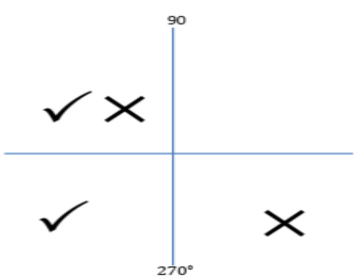
$$7tan\ \theta\ = -3$$

$$tan\ \theta\ = \frac{-3}{7}$$

→use conditions to identify the quadrants and tick them with different symbols on the Cartesian plane.

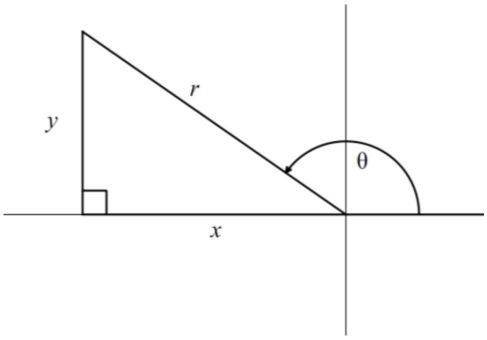
tan is negative in the 2nd and 4th quadrants.

This interval represents 2nd and 3rd quadrants



Given that $tan\ \theta = \frac{-3}{7}$ and θ ∈ [90°: 270°, Where X represents the quadrant, we obtained from tan and tick is the quadrants obtained from the interval. We can notice the 2nd quadrant satisfies both conditions, hence the terminal arm will be in the 2nd quadrant.

→ Complete the right-angled triangle.



For some trigonometry problems, it is helpful to draw a diagram showing the angle involved and the *x, y* and *r* values. It is advisable to find all the values of *x, y* and *r* before solving the trigonometry problems that will need the right-angled triangle: $x^2+y^2 = r^2$ where r is the hypotenuse.

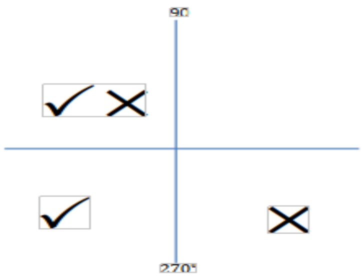
Remember hypotenuse is ALWAYS positive, and for the values of *x* and *y*, they can be positive or negative depending on the quadrant you are working in.

Worked example 2

Given that $\sin\theta = \frac{3}{5}$, and θ ∈ [90°: 360°,] find the value of the following ***without the use of a calculator.***

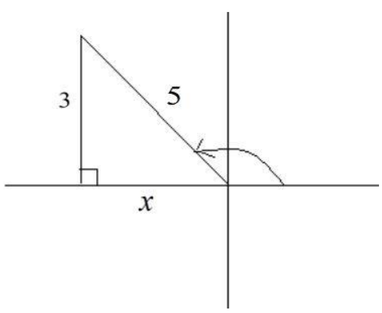
without a calculator – It means you must use the values from the TRIANGLE

Solutions:



We are going to draw the triangle in a 2nd quadrant.

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{3}{5} = \frac{y}{r},$$



In the triangle above we do not have the value of x, then we will calculate it using **theorem of** Pythagoras:

$x^2 + y^2 = r^2$

$x^2 + (3)^2 = (5)^2$

$x^2 + 9 = 25$

$x^2 = 25 - 9$

$x^2 = 16$

$x = \sqrt{16}$

$x = \pm 4$

$x = -4$, because the x value in the 2nd quadrant is negative.

We can answer the following questions using the triangle above.

a. $cos\ \theta$
b. $tan(\ 180° - \theta)$
c. $sin(-\theta)$
d. $sin\ (\theta - 180°)$
e. $cos\ (450° + 2\theta)$
f. $sin\ (\ 45° - \theta\ )$

Solutions

a. $cos\ \theta = \frac{opposite}{hypotenuse}$

$= \frac{-4}{5}$

$= -\frac{4}{5}$

b. $tan(\ 180° - \theta)$

$= -\tan\theta = \frac{opposite}{adjacent}$

$= -(\frac{3}{-4})$

$= \frac{3}{4}$

c. $sin(-\theta)$

$= -\sin\theta$

$= -\frac{3}{5}$

d. $sin\ (\theta - 180°)$

$= \sin(-180° + \theta\ )$

$= sin\ [-(\ 180° - \theta)]$

$= \quad -sin\,(\,180° \,-\, \theta)$

$= \quad -sin\,\theta$

$= \; -\,\frac{3}{5}$

e. $\quad cos\,(450° \,+\, 2\theta)$

$= \quad cos\,(450° - 360° \,+\, 2\theta)$

$= \quad cos\,(90° \,+\, 2\theta)$

$= \; -\sin 2\theta$

$= \; -2sin\theta.cos\theta$

$= \; -2(\frac{3}{5}).(-\frac{4}{5})$

$= \; \frac{24}{25}$

f. $\quad sin\,(\,45° \,-\, \theta\,)$

$= sin\,(45° \,-\, \theta)$

$= sin\,45°.cos\,\theta° \,-\, cos\,45°.sin\,\theta°$

$$= \frac{1}{\sqrt{2}} . -\frac{4}{5} - \frac{1}{\sqrt{2}} . \frac{3}{5}$$

$$= \frac{4}{5\sqrt{2}} - \frac{3}{5\sqrt{2}}$$

$$= \frac{1}{5\sqrt{2}}$$

$$= \frac{1}{5\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{10}$$

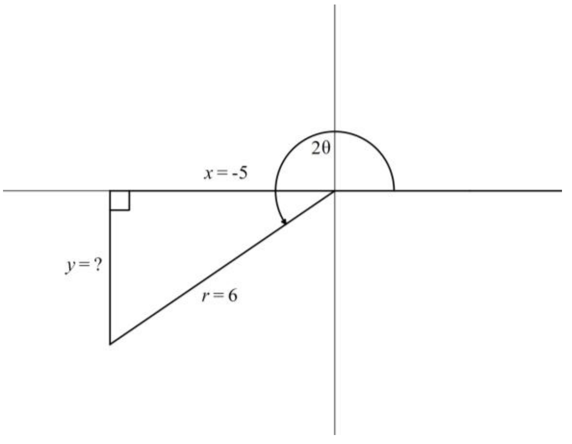
Worked example 3

If $\cos 2\theta = -\frac{5}{6}$, where $2\theta \in [180°;270°]$, calculate, **without using a calculator**, the values in simplest form of:

1. $\sin\ 2\theta$
2. $\sin^2\theta$

Solutions

2θ is in 3rd quadrant because cos is negative, and the interval represents the 3rd quadrant.



Before we can answer the questions, we can note that in the triangle we do not have y-value.

$$x^2 + y^2 = r^2$$
$$(-5)^2 + y^2 = (6)^2$$
$$y^2 = 36 - 25$$
$$y^2 = 11$$
$$\mathrm{y} = \sqrt{11}$$
$$y = \pm\sqrt{11}$$

$y = -\sqrt{11}$, because the y value in the 3rd quadrant is negative.

a. $Sin\ 2\theta = \frac{-\sqrt{11}}{6}$

b. $cos2\theta = -\frac{5}{6}$

$$-\frac{5}{6} = 1 - 2\sin^2\theta$$

$$2\ sin^2\theta = 1 + \frac{5}{6}$$

$$2\ sin^2\theta = \frac{11}{6}$$

$$sin^2\theta = \frac{11}{6} \div 2$$

$$sin^2\theta = \frac{11}{12}$$

Worked example 4

If cos 42° $=$ p find the value of the following in terms of *p.*

a. Calculate the value of *x, y* and *r*.

b. Indicate all angles in a triangle

c. $sin\ 42°$

d. $cos\ 138°$

e. $tan\ 312°$

f. $sin\ 132°$

g. $cos\ (-48°)$

h. $sin\ 84°$

i. $sin\ 402°$

j. $cos\ 768°$

k. $2\ sin\ 21°. cos\ 21°$

l. $\cos^2 24° - \sin^2 24°$

m. $1\ -\ 2sin^2\ 24°$

n. $sin\ 21°. cos\ 21°$

o. $cos\ 21°$

p. $sin\ 21°$

Solutions

a. $\sin 42° = \frac{p}{1} = \frac{y}{r}$

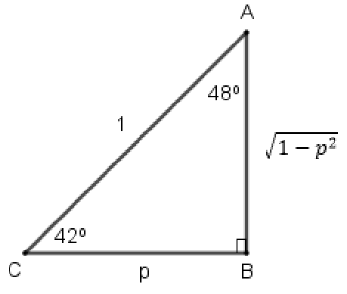
$x^2 + p^2 = 1^2$

$x^2 = 1 - p^2$

$x = \pm\sqrt{1-p^2}$

$x = -\sqrt{1-p^2}$

b.



c. $\sin 42°$

$= \frac{\sqrt{1-p^2}}{1}$

$= \sqrt{1-\ p^2}$

d. cos 138°

$= \quad cos\,(180 - 42°)$

$= \quad -cos\,42°$

$= \quad -p$

e. tan 312°

$= \quad tan\,(360° - 48°)$

$= \quad -tan\,48°$

$= \quad -\frac{p}{\sqrt{1-p^2}}$ (Rationalize)

$= \frac{-p}{\sqrt{1-p^2}} \times \frac{\sqrt{1-p^2}}{\sqrt{1-p^2}}$

$= \frac{-p\sqrt{1-p^2}}{1-p^2}$

f. sin 132°

$= \quad sin\,(180° - 48°)$

$= \quad sin\,48°$

$= \quad p$

g. $cos\,(-48°)$

$= \quad cos\,48°$

$= \quad \sqrt{1-p^2}$

h. sin 84°

$= \quad sin\,2\,(42°)$

$= \quad 2sin\,42\,cos\,42°$

$= \quad 2.\sqrt{1-p^2}\;.p$

$= \quad 2p.\sqrt{1-p^2}$

i. sin 402°

$= \quad sin\,(402° - 360°)$

$= \quad sin\,42°$

$= \quad \sqrt{1-p^2}$

j. cos 768°

$= \quad cos\,(768° - 360° - 360°)$

$= \quad cos\,48°$

$= \quad \sqrt{1-p^2}$

k. 2 sin 21°.cos 21°

$= \quad sin\,2(21°)$

$= \quad sin\ 42°$

$= \quad \sqrt{1 - p^2}$

l. $\cos^2 24° - \sin^2 24°$

$= \quad cos\ 2(24°)$

$= \quad cos\ 48°$

$= \quad \sqrt{1 - p^2}$

m. $1 - 2\sin^2 21°$

$= \quad cos\ 2(21°)$

$= \quad \cos 42°$

$= \quad p$

n. $sin\ 21°. cos\ 21°$

$= \quad (\frac{1}{2} \times 2 \sin 21°. \cos 21°)$ Remember: $\frac{1}{2} \times 2 = 1$

$= \quad \frac{1}{2}\ sin2\ (21°)$

$= \quad \frac{1}{2}\ sin\ 42°$

$= \quad \frac{1}{2}\ \sqrt{1 - p^2}$

o. $cos\ 21°$

$cos\ 42° = cos\ 2(21°) = 2cos^2 21° - 1$

$p = 2\cos^2 21° - 1$

$p + 1 = cos^2 21°$

$\frac{1}{2}(p + 1) = cos^2 21°$

$\sqrt{\frac{1}{2}(p + 1)} = cos\ 21°$

p. $\sin 21°$

$\cos 42° = \cos 2(21°) = 1 - 2\sin^2 21°$

$p = 1 - 2\sin^2 21°$

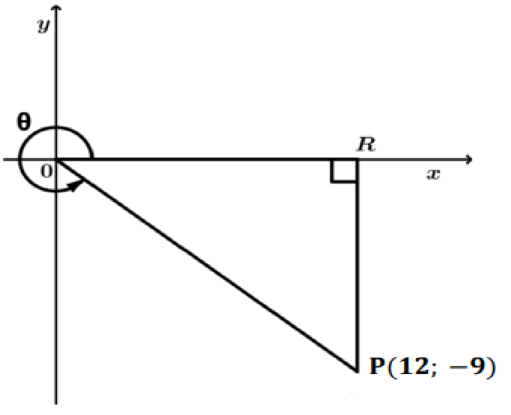
$2\sin^2 21° = 1 - p$

$\sin^2 21° = \frac{1}{2}(1 - p)$

$\sin 21° = \sqrt{\frac{1}{2}(1 - p)}$

Worked example 5

5.1 In the diagram below, P(12; −9) is a point in the Cartesian plane. Reflex angle $R\hat{O}P$ is equal to θ.



Calculate, **without using a calculator**, the:

5.1.1 Length of OP

5.1.2 The value of:

(a) $\cos(90° - \theta)$

(b) $\sin(\theta + 30°)$

Solutions

5.1.1	$OP = \sqrt{144 + 81}$ $= \sqrt{225}$ $= 15$
5.1.2 (a)	$\sin\theta = -\frac{9}{15} = -\frac{3}{5}$
5.1.2 (b)	$\sin(\theta + 30°) = \sin\theta\cos 30° + \cos\theta\sin 30°$ $= \left(-\frac{3}{5}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{12}{15}\right)\left(\frac{1}{2}\right)$ $= -\frac{3\sqrt{3}}{10} + \frac{6}{15}$ $= \frac{4 - 3\sqrt{3}}{10}$

Trigonometry Identity

Trigonometry Identity – an equality which is true for all values of an unknown variable, for which both sides of the identity are defined (so no zero denominators). It differs from a trigonometric equation which is true only for certain values of the unknown variable.

- To prove an identity, you must prove LHS = RHS

a) Square Identity **b) Quotient identity**

$$sin^2\,\theta + cos^2\,\theta = 1$$

$$sin^2\,\theta = 1 - cos^2\,\theta$$

$$cos^2\,\theta = 1 - sin^2\,\theta$$

$$tan\,\theta = \frac{sin\,\theta}{cos\,\theta}$$

$$tan\,2\,\theta = \frac{sin\,2\,\theta}{cos\,2\,\theta}$$

$$tan(\,\theta + \alpha) = \frac{sin(\,\theta + \alpha)}{cos(\,\theta + \alpha)}$$

Worked example 1

Simplify the expression using the identities proved before:

a. $(1 - sin\theta)\,(1 + sin\theta)$

b. $\cos^2\theta - 1$

c. $\tan^2\theta - \tan^2\theta\,.\sin^2\theta$

Solutions:

(a) $(1-\sin\theta)(1+\sin\theta)$

$= 1-\sin^2\theta$

$= \cos^2\theta$ (Use the identity $1-\sin^2\theta = \cos^2\theta$)

(b) $\cos^2\theta - 1 = -\sin^2\theta$ (By manipulating $\sin^2\theta + \cos^2\theta = 1$ we can deduce this) OR

$\cos^2\theta - 1 = -(1-\cos^2\theta)$ Factorise (or Change of sign rule)

$= -\sin^2\theta$ Use the identity: $\sin^2\theta = 1-\cos^2\theta$

(c) $\tan^2\theta - \tan^2\theta.\sin^2\theta$

$= \tan^2\theta(1-\sin^2\theta)$ Factorise by taking out the HCF

$= \dfrac{\sin^2\theta}{\cos^2\theta}(\cos^2\theta)$ If $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ then $\tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}$

$= \dfrac{\sin^2\theta}{\cos^2\theta}\left(\dfrac{\cos^2\theta}{1}\right)$

$= \sin^2\theta$

c) Compound angles

When two angles are added or subtracted to form a new angle, then a **Compound** or **a double angle** is formed.

$$\boxed{\begin{aligned} sin(A+B) &= \sin A \cos B + \cos A \sin B \\ sin(A-B) &= \sin A \cos B - \cos A \sin B \\ cos(A+B) &= \cos A \cos B - \sin A \sin B \\ cos(A-B) &= \cos A \cos B + \sin A \sin B \end{aligned}}$$

To derive the other formulae the formula for $cos(\alpha - \beta)$ is used.

Derivation of $\boldsymbol{cos(\alpha+\beta) = cos\alpha\ cos\beta - sin\alpha\ sin\beta}$:

$cos(\alpha+\beta) = cos(\alpha-(-\beta))$

$cos(\alpha+\beta) = cos\alpha.cos(-\beta) + sin\alpha.sin(-\beta)$

$but\ cos(-\beta) = cos\beta\ and\ sin(-\beta) = -sin\beta$

$\therefore cos(\alpha+\beta) = cos\alpha.cos\beta - sin\alpha.sin\beta$

Derivation of $\boldsymbol{sin(\alpha+\beta) = sin\alpha\ cos\beta + cos\alpha\ sin\beta}$:

$sin(\alpha+\beta) = cos\ (90° - (\alpha+\beta))$

$sin(\alpha+\beta) = cos((90°-\alpha) - \beta)$

$sin(\alpha+\beta) = cos(90°-\alpha).cos\beta + sin(90°-\alpha)\ sin\beta$

$but\ cos(90°-\alpha) = sin\alpha\ and\ sin(90°-\alpha) = cos\alpha$

$\therefore sin(\alpha+\beta) = sin\alpha.cos\beta + cos\alpha.sin\beta$

Derivation of $\boldsymbol{sin(\alpha-\beta) = sin\alpha\ cos\beta - cos\alpha\ sin\beta}$:

$sin(\alpha-\beta) = sin(\alpha+(-\beta))$

$sin(\alpha-\beta) = sin\alpha.cos(-\beta) + cos\alpha.sin(-\beta)$

$but\ cos(-\beta) = cos\beta\ and\ sin(-\beta) = -sin\beta$

$\therefore sin(\alpha-\beta) = sin\alpha.cos\beta - cos\alpha.sin\beta$

Worked example 2

a. Find the value of $sin\ (45° + 30°)$ without the use of a calculator.

b. Expand the following using the compound angle formula $cos(60° - A)$.

c. Find the value of $cos\ 15°$ without using a calculator.

Solutions

a. $sin\ (30° + 45°)$

$= sin\ 30°.cos\ 45° + cos\ 30°.sin\ 45°$

$$= \frac{1}{2}.\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}+\sqrt{6}}{4}$$

b. $cos(60° - A)$.

$$= cos\,A.cos60° + sin\,A.sin60°$$

$$= cos\,A.\frac{1}{2} + sin\,A.\frac{\sqrt{3}}{2}$$

$$= \frac{1}{2}(cos\,A + \sqrt{3}\,sin\,A)$$

c. $sin\,15°$

$$= sin\,(45° - 30°)$$

$$= sin\,45°.cos\,30° - cos\,45°.sin\,30°$$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

Worked example 3

Simplify without the use of a calculator:

a. $cos\,70°.cos\,10° + sin\,70°.sin10°$
b. $sin\,20°.cos\,40° + cos\,20°.sin\,40°$
c. $cos\,50°cos\,40° - sin\,50°sin\,40°$
d. $cos\,3\beta.cos\,\beta + sin\,3\beta.sin\,\beta$
e. $cos\,3\theta.sin\theta + sin\,3\theta.cos\,\theta$
f. $sin\,10°sin\,20° - cos\,20°\,cos\,10°$

Solutions

$a.$ $cos\,70°.cos\,10° + sin\,70°.sin10°$

$$= cos\,70°\,cos\,10° + sin70°\,sin10°$$

$$= cos(70° - 10°)$$

$$= cos\,60°$$

$$= \frac{1}{2}$$

$b.$ $sin\,20°cos\,40° + cos\,20°sin\,40°$

$= sin(20° + 40°)$

$= sin60°$

$= \frac{\sqrt{3}}{2}$

$c.\quad cos\ 50° cos\ 40° - sin\ 50° sin\ 40°$

$= \cos(50° + 40°)$

$= \cos 90°$

$= 0$

$d.\quad cos\ 3\beta . cos\ \beta\ + sin\ 3\beta . sin\ \beta$

$= cos(3\beta\ -\ \beta)$

$= cos\ 2\ \beta$

$e.\quad sin\ \theta\ cos3\theta\ + cos\ \theta\ sin3\theta$

$= sin(\theta\ -\ 3\theta)$

$= sin(-2\theta)$

$= -sin2\theta$

$f.\quad sin\ 20°. sin10° - \ cos\ 20°. cos\ 10°$

$= \ - cos20°\ cos10°\ +\ sin20°\ sin10°$

$= \ -(cos20°\ cos10°\ \ -\ \ sin20°\ sin10°)$

$= - cos(20°\ +\ 10°)$

$= -\ cos\ 30°$

$= \ -\ \frac{\sqrt{3}}{2}$

<u>D) Double angles</u>

$sin\ 2\ A = 2\ sin\ A\ cos\ A$

$cos\ 2\ A = cos^2\ A - sin^2\ A$

$= 1 - 2\ sin^2\ A$

$= 2\ cos^2\ A - 1$

The double angles are derived from the fact that $2\theta = \theta + \theta$

Derivation of $\boldsymbol{sin\ 2\theta\ = 2sin\ \theta . cos\ \theta}$

$sin\ 2\ \theta\ = \ sin(\theta + \theta)$

$= \ sin\theta . cos\theta\ +\ cos\theta . sin\theta$

$= \ 2sin\theta . cos\theta$

Derivation of $\boldsymbol{cos2\theta\ = cos^2\ \theta - sin^2\ \theta}$

$cos\ 2\theta\ = \ cos\ (\theta + \theta)$

$= \ cos\theta\ . cos\theta\ -\ sin\theta . sin\theta$

$= cos^2 \theta - sin^2 \theta$

Derivation of $\boldsymbol{cos2\theta \ = 2\, cos^2 \theta - \ 1}$

$cos\ 2\theta \ = \ cos\ (\theta + \theta)$

$= \ cos\theta\ .cos\theta \ - \ sin\theta. sin\theta$

$= cos^2 \theta - sin^2 \theta$

$= cos^2 \theta - (1 - cos^2 \theta)$

$= 2cos^2 \theta - 1$

Derivation of $\boldsymbol{cos2\theta \ = 1 - 2sin^2 \theta}$

$cos\ 2\theta \ = \ cos\ (\theta + \theta)$

$= \ cos\theta\ .cos\theta \ - \ sin\theta. sin\theta$

$= cos^2 \theta - sin^2 \theta$

$= 1 - sin^2 \theta - sin^2 \theta$

$= 1 - 2sin^2 \theta$

Worked example 4

1. Use the **double angle formulae of sine** to expand the following:

$(a)\ \ sin\ 6\beta$

$(b)\ \ sin\ 70°$

$(c)\ \ sin\ 10\theta$

2. Use the **double angle formulae of cosine** to expand the following:

$(a)\ \ cos\ 14\beta$

$(b)\ \ cos\ 50°$

$(c)\ \ cos\ 3\theta$

Solution

$1.\ (a) \qquad sin\ 6\beta$

$= sin[2\ (\boldsymbol{3\beta})\ ]$

$= \ 2\ sin\ 3\beta . cos\ 3\beta$

$(b) \qquad sin\ 70$

$= sin[2\ (\boldsymbol{35}°)\ ]$

$= \ 2\ sin\ 35°. cos\ 35°$

$(c) \qquad sin\ 8\theta$

$= sin[2\ (\boldsymbol{4\theta})\ ]$

$= \ 2\ sin\ 4\theta . cos\ 4\theta$

$2. (a)\ \ cos 14\beta$

$= \ cos\ [2\ (\boldsymbol{7\beta})\ ]$

$= cos^2 7\beta - sin^2 7\beta$

$(b) \cos 50°$

$= \quad cos\ [2\ (25)\ ]$

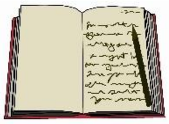
$= 2cos^2 25 - 1$

$(c)\ \ cos\ 3\theta$

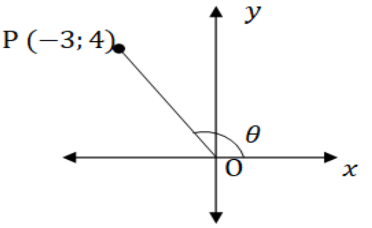
$= \ cos\ [2\ (\frac{3}{2}\theta)\ ]$ $\qquad$ $NB: 2 \times \frac{3}{2} = 3$

$= 1 - 2sin^2 \left(\frac{3}{2}\theta\right)$

Activities



1. Use the following diagram to answer the questions below.



1.1 the length of OP.

1.2 $tan\theta$

1.3 $\sin{(180° - \theta)}$

1.4 $\cos{(180° + \theta)}$

1.5 $sin2\theta$

1.6 $cos2\theta$

1.7 $\sin{(30° + \theta)}$

1.8 $\cos{(45° - \theta)}$

2. If $5\cos x = 4$ and $x \in [90°; 360°]$, determine the value of $3\sin 2x$, without using a calculator.

3. If $13\cos A = 5$ and $\tan B = \frac{3}{4}$, $A \in [0°; 270°]$ and $B \in [0°; 180°]$.

Determine, without using the calculator,:

3.1 $\sin A$

3.1 $\sin{(A + B)}$

4. 3 If $sin\ 27° = m$, express each of the following in terms of q.

4.1 $\sin 117°$

4.2 $tan\ (-27°)$

4.3 $\cos^2 63°$

4.4 $sin\ 72°$

4.5 $cos\ 54°$

5. Given $\sin 34° = p$, find the value of $sin17°cos17°$ in terms of p.

6. If $cos\ 73° cos\ 31° + sin\ 73° Sin\ 31° = \rho$, then determine the value of $\cos^2 21° - \sin^2 21° + 7$.

SECTION 3: Reduction Formula

Reduction formula

Content and worked examples:

Reduction formulae are used to reduce the trigonometric ratio of any angle to the trigonometric ratio of an acute angle.

Approach to simplify exercises:

- ✓ Identify the quadrant to which the angle lie.
- ✓ If you use the reduction formula of $(90^0 \pm \theta)$, remember to change to co-function
- ✓ Write the correct trig ratio together with the acute angle.
- ✓ Take **one factor at a time** and make sense/simplify it.

  Note:

  - Angles greater than (360^0)
  - Negative angles
  - Complementary angles.

90°

2nd Quadrant

$\sin(180° - A) = \sin A$

$\cos(180° - A) = -\cos A$

$\tan(180° - A) = -\tan A$

$\sin(90° + A) = \cos A$

$\cos(90° + A) = -\sin A$

1st Quadrant

$\sin(90° - A) = \cos A$

$\cos(90° - A) = \sin A$

$\sin(360° + A) = \sin A$

$\cos(360° + A) = \cos A$

$\tan(360° + A) = \tan A$

180° 360°

3rd Quadrant

$\sin(180° + A) = -\sin A$

$\cos(180° + A) = -\cos A$

$\tan(180° + A) = \tan A$

4th Quadrant

$\sin(360° - A) = -\sin A$

$\cos(360° - A) = \cos A$

$\tan(360° - A) = -\tan A$

270°

Worked example 1

Simplify as far as possible without the use of a calculator.

a. $\frac{\cos(180°+\theta)}{\cos(180°-\theta)}$

b. $\tan(90+\theta).\sin(180+\theta)$

c. $\sin 120°$

d. $\cos 260°$

e. $\frac{\tan(180°-\theta).\sin(360°+\theta)}{\tan(360°+\theta)}$

f. $\frac{\cos(360°-x).\sin(180°+x)}{\cos(180°+x)}$

Solution

a. $\frac{\cos(180°+\theta)}{\cos(180°-\theta)}$

$= \frac{\cos(\theta)}{-\cos(\theta)}$ (In 3rd quadrant and cosine is negative and 2nd quadrant and cosine is negative in the 2nd)

$= -1$

b. $\tan(90+\theta).\sin(180+\theta)$

$= \frac{\sin(90+\theta)}{\cos(90+\theta)} \times -\sin(\theta)$ (Change tan to $\frac{sin}{cos}$)

$= \frac{\cos(\theta)}{-\sin(\theta)} \times -\sin(\theta)$

$= \cos(\theta)$

c. $\sin 120°$ (Reduce obtuse angle to be acute angle)

$= \sin(180° - 60°)$

$= \sin 60°$

$= \frac{\sqrt{3}}{2}$

d. $cos 260°$ (Reduce obtuse angle to be acute angle)

$= \cos(180° + 80°)$

$= -cos\ 80°$

e. $\frac{\tan(180°-\theta).\sin(360°+\theta)}{\tan(360°+\theta)}$

$$\frac{\overset{3^{rd}}{\tan(180°-\theta)}.\overset{4^{th}}{\sin(360°+\theta)}}{\underset{4^{th}}{\tan(360°+\theta)}}$$

$= \frac{(-\tan\theta)(+\sin\theta)}{\tan\theta}$ (Simplify)

$= -\sin\theta$

f. $\frac{\cos(360°-x).\sin(180°+x)}{\cos(180°+x)}$

$= \frac{(+\cos(x)).(-\sin(x))}{(-\cos(x))}$

$= \frac{-\sin(x)}{-1}$

$= \sin(x)$

Worked example 2

$a.\ sin(540° + \theta)$ $b.\ cos(720° - \theta)$

Solutions

a. $sin(540° + \theta)$

$= sin(540° + \theta - 360°)$

$= sin(180° + \theta)$

$= -sin\,\theta$

b. $cos(720° - \theta)$

$= cos(720° - \theta - 360°)$

$= cos(360 - \theta)$

$= cos\,\theta$

Negative angles

It is important to note that on the Cartesian plane an angle is considered to be positive if the rotation is in an anti-clockwise direction and negative if the rotation is clockwise.

$\boldsymbol{sin(-\theta) = -sin\,\theta}$
$\boldsymbol{cos(-\theta) = cos\,(\theta)}$
$\boldsymbol{tan(-\theta) = -tan\,\theta}$

Worked example 3

Write the following as a function value of a positive acute angle.

$a.\ sin\,(-50°)$ b. $\tan(150°)$ c. $cos\,(-35°)$

Solutions

a. $\sin(-50°)$

$= -sin\,50°$

b. $\tan(-150$

$= -tan\,150°$

$= -\tan(180° - 30°)$

$= -(-\tan 30°)$

$= tan30°$

c. $cos\,(-35°)$

$= cos\,35°$

Worked example 4

Remember:

a + b = b + a

-a + b = b – a

-a + b = - (a – b)

-a – b = - (a + b)

Simplify the following to a function value of θ:

$a.\ \ tan\ (\theta\ - 180°)$

$b.\ \ sin\ (-\ \theta\ -\ 360°)$

Solutions

a. $\tan(\theta - 180°)$

$=\ tan\ (-180° + \theta)$ [*tan (− angle) = − tan (angle)*]

$= -\ tan\ [-(180° - \theta)]$

$=\ -\ tan\ (180° - \theta)$

$=\ tan\ \theta$

b. $sin\ (-\ \theta\ -\ 360°)$ [take out common sign]

$=\ sin\ [-(\theta\ +\ 360°)]$

$=\ -sin\ (\theta\ +\ 360)\ °$ [*sin(− angle) = − sin (angle)*]

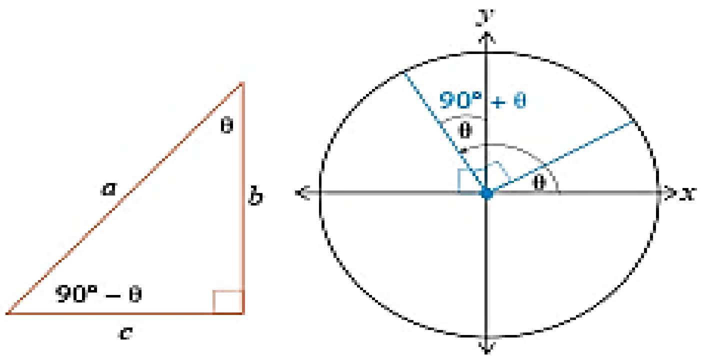
$=\ -sin\ (360° + \theta)$

$=\ -[-sin(\theta)]$

$=\ sin(\theta)$

The co-ratios are $90° - \theta$ and $90° + \theta$

Co-ratios are also called complementary ratios.



Look at the figure and you will see that :

$\sin(90° - \theta) = \frac{b}{a} = cos\theta$ and $\cos(90° - \theta) = sin\theta$

The $(90° + \theta)$ lies in Quadrant 2.

$\sin(90° + \theta) = \frac{b}{a}\cos\theta$ and $\cos(90° + \theta) = -sin\theta$, because $cos\theta$ is negative in Quadrant 2.

The following reduction formulae are used to reduce the co-functions:

$\sin(90° - \theta) = cos\theta$ $\cos(90° - \theta) = sin\theta$	$\sin(90° + \theta) = cos\theta$ $\cos(90° + \theta) = -sin\theta$

Worked example 5

(a) $\frac{sin\,(90°-\theta).sin\theta}{cos\theta.cos(90°+\theta)}$ (b) $sin(\theta - 90°)$ (c) $cos\ (\theta - 90°)$

Solution

(a) $\frac{sin\,(90°-\theta).sin\theta}{cos\theta.cos(90°+\theta)}$ $= \frac{cos\theta.sin\theta}{cos\theta.-sin\theta}$ $= -1$	(b) $sin(\theta\ - 90°)$ $= sin\ [-(\theta\ + 90°)]$ $= -sin(\theta\ +\ 90)$ $= -sin(90\ +\ \theta)$ $= -\ cos\ \theta$	(c) $cos\ (\theta - 90°)$ $=\ cos\ [-(\theta\ + 90°)]$ $= \cos(\theta\ +\ 90)$ $= cos\ (90\ +\ \theta)$ $= -sin\ \theta$

Worked example 6

a. Reduce $\frac{\tan 43°.\sin 47°.2\cos 137°}{2\cos 317°.\sin 133°-1}$ to a single trigonometric ratio of one angle without the use of a calculator .

b. Show that $Sin(B + A) - Sin(B - A) = 2sinAcosA$

Solutions

a $\quad \frac{\tan 43°\,.\,\sin 47°\,.\,2\cos 137°}{2\cos 317°\sin 133° - 1}$

$\frac{\left(\frac{\sin 43°}{\cos 43°}\right)(\cos 43°)(-2\cos 43°)}{2\cos 43°\,.\,\cos 43° - 1}$

$\frac{\sin 43°\,(-2\cos 43°)}{\cos 2(43°)}$

$\frac{-\sin 2(43°)}{\cos 86°}$

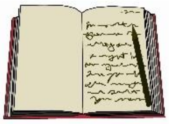
$-\tan 86°$

b $\quad \sin(B + A) - \sin(B - A)$

$= \sin B\cos A + \cos B\sin A - (sinB\cos A - \cos B\sin A)$

$= \cos B\sin A + \cos B\sin A$

$= 2\sin A\cos B$



Activities:

1. Simplify the following **without using a calculator** to a single trigonometric ratio:

1.1 $\dfrac{sin(90^0+x).tan(360^0+x)}{sin(180^0+x).cos(90-x^0)+cos(540^0+x).cos(-x)}$

1.2 $\dfrac{cos(720^0-x).sin(360^0+x).tan(x-180^0)}{sin(-x).cos(90^0-x)}$

1.3 $\dfrac{cos(-210).sin^2 405^0 \, cos\, 14^0}{tan\, 120^0.sin\, 104^0}$

1.4 $\dfrac{sin(450^0-x)\, tan(x-180^0)\, sin\, 23^0.cos\, 23^0}{cos\, 46^0.sin(-x)}$

1.5 $\dfrac{4\, cos(-x).\cos(90°+x)}{\sin(30°-x).cos\, x+cos(30°-x).sin\, x}$

2.1 **WITHOUT using a calculator**, determine the following in terms of sin 25°:

2.1.1 sin 335°

2.1.2 cos 50°

2.2 Simplify the following expression to ONE trigonometric ratio:

$$\frac{sin(-2x).(1-sin^2 x)}{sin(90°+x).tan\, x}$$

2.3 **WITHOUT using a calculator**, simplify $(p\, tan\, 30° + q\, sin\, 60°)^2$.

2.4 Given: $cos(A-B) = cos\, A\, .\, cos\, B + sin\, A\, .\, sin\, B$

2.4.1 Use the formula for $cos(A-B)$ to derive a formula for $sin(A-B)$.

2.4.2 Prove that $sin\, 9\, A + sin\, A = 2\, sin\, 5\, A.\, cos\, 4\, A$.

2.4.3 Write down the maximum value of $3^{2\, sin\, 5A\, cos\, 4A}$

3. Evaluate: $sin\, 70°\, cos\, 40° - cos\, 70°\, sin\, 40°$

4. Evaluate each of the following without using a calculator.
   - 4.1 Sin 75°
   - 4.2 cos 15°
   - 4.3 cos 105°
   - 4.4 sin 165°
   - 4.5 sin 36°cos 54°+cos 36°sin 54°
   - 4.6 cos 42°cos 18° − sin 42°sin 18°
   - 4.7 sin 85°sin 25°+cos 85°cos 25°
   - 4.8 sin 70°cos 40° − cos 70°sin 40°

5 Prove the identity:

$$\frac{3\,sin\,x + 2\,sin\,2\,x}{2 + 3\,cos\,x + 2\,cos\,2\,x} = tan\,x$$

6. **Without the use of a calculator**, simplify the following expression to ONE trigonometric ratio:

$$2\,cos^2\,1\,5^\circ - 1 + \frac{2\,sin\,1\,40^\circ}{cos\,3\,10^\circ}$$

7 Simplify:

$$\frac{1 - sin(-\theta)\,cos(90^\circ + \theta)}{cos(\theta - 360^\circ)}$$

8 Consider: $\frac{cos\,2x + sin\,2x - cos^2\,x}{sin\,x - 2\,cos\,x} = -\,sin\,x$

8.1 Prove the above identity.

8.2 Determine the value of $\frac{cos\,2x + sin\,2x - cos^2\,x}{-3\,sin^2\,x + 6\,sin\,x\,cos\,x}$

9. Prove the following identity : $\frac{\sin 2x - cos\,x}{1 - cos\,2x - sin\,x} = \frac{1}{tan\,x}$

SECTION 4: Trigonometry Functions

In this section, we will discuss the graphs of the basic functions $y = sin\,\theta$ and $y\ =\ cos\,\theta$ and then introduce the general functions:

$y = asink(x - p) + q$, $y = a\cos k(x - p) + qy = a\,tan\,k\,(x - p) + q$

Worked example 1.

On the same system of axes, draw the sketch graphs of:

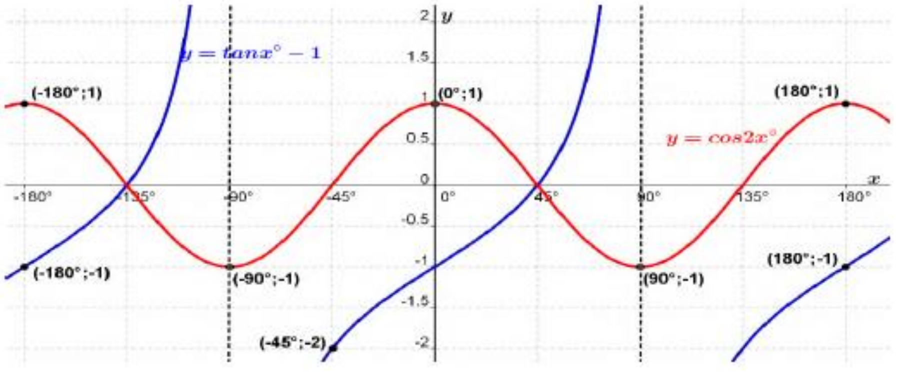
$f(x) = \tan x - 1$ and $g(x) = \cos 2x$, for the interval $[-180°; 0°]$. Show all the intercepts with the axes and the co-ordinates of the turning points. Show the asymptotes of $y = \tan x - 1$.

(a) Use the sketch graphs to determine the value of x if: $\cos 2x + 1 \leq \tan x$,in the interval $[-180°; 0°]$.

(b) If the curve of $y = \tan x - 1$ is moved upwards by 3 units, what will the new equation be?

(c) Write down the period of $y = \cos 2x$.

Solution



(a) $\cos 2x + 1 \leq \tan x$

$\cos 2x \leq \tan x - 1$

$g(x) \leq f(x)$

$-135° \leq x < -90°$ and $-90° < x \leq 135°$

(b) $y = tan\ x + 2$

(c) period :180°

Worked example 2

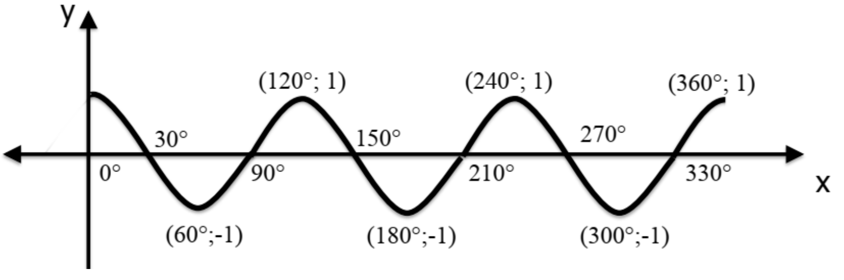
Find the period of f and draw the graphs of f:

a. $f(x) = \cos 3x$, $x \in [0°:360°]$

b . $f(x) = \tan\frac{1}{2}x$, $x \in [0°;360°]$

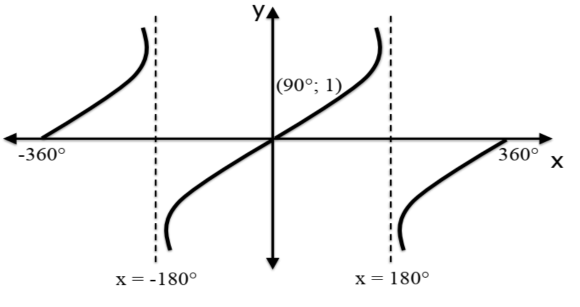
Solution

a. Period: $\frac{360°}{k} = \frac{360°}{3} = 120°$



We have three full waves of sine in the question.

b. Period: $\frac{180°}{k} = \frac{180°}{\frac{1}{2}} = 360°$



HORIZONTAL SHIFT

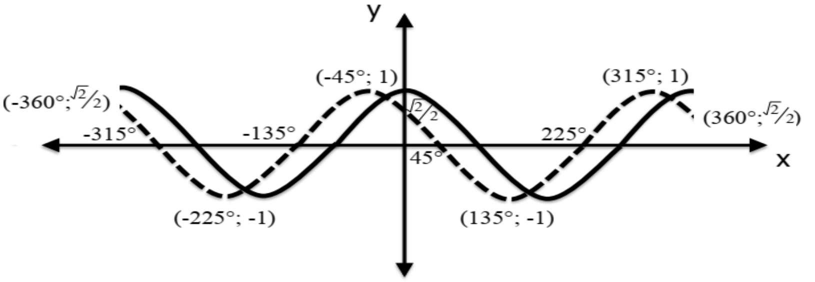
- $y = \sin(x - p)$ or $y = \cos(x - p)$ or $y = \tan(x - p)$

If p > 0 : shift right (e.g: $y = \sin(x - 30°)$)
p < 0 : shift left (e.g: $y = \cos(x + 45)$)

How to plot a horizontal shift:
- Plot the original curve
- Move the critical points left/right
- Label the x-cuts and turning points
- Calculate and label the endpoints and y-cut

Worked example 3

1. $y = \cos(x + 45°)$ for $x \in [-360°; 360°]$ (dotted line)
$y = \cos x$ (solid line - for comparison)

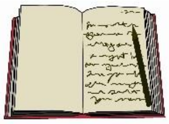


Endpoints:

$\cos(-360° + 45°) = \frac{\sqrt{2}}{2}$ and $cos(-360° + 45°) = \frac{\sqrt{2}}{2}$

y-cut:

$\cos(0° + 45°) = \frac{\sqrt{2}}{2}$

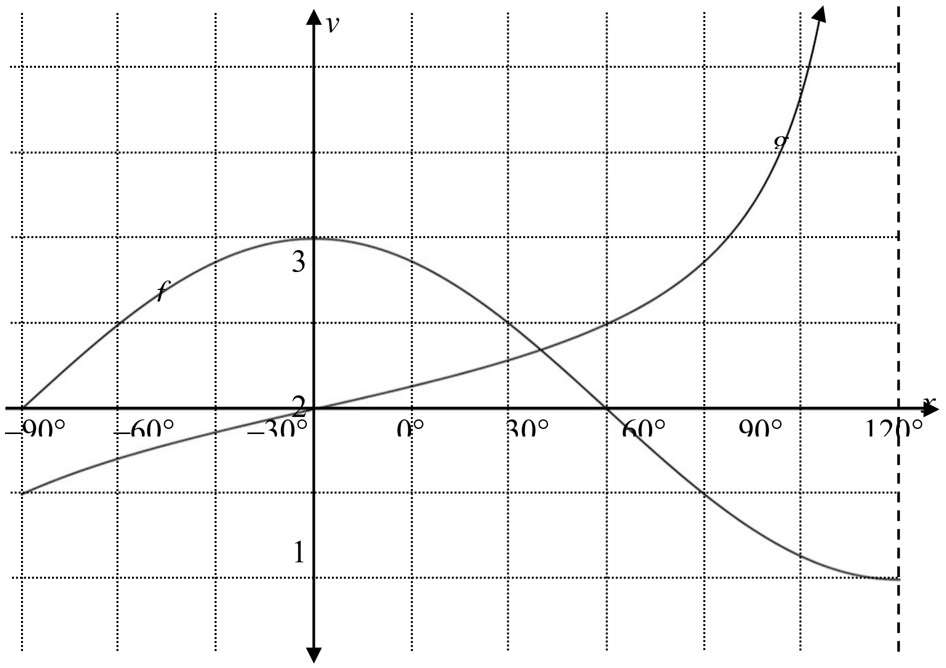


Trigonometry Functions

Activities:

1.

In the diagram below, the graphs of $f(x) = 2\cos x$ and $g(x) = \tan bx$ are drawn for the interval $x \in [-90° ; 120°]$.



Use the graphs to answer the following questions.

1.1 Write down the value of b.

1.2 Write down the range of g for the interval $x \in [-90°; 120°]$.

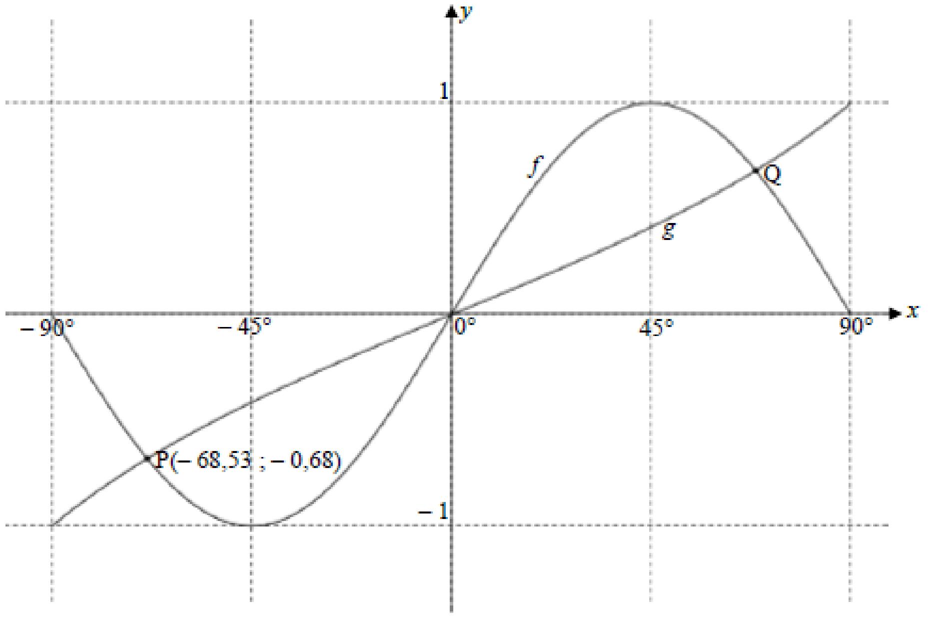
1.3 Write down the period of g.

1.4 Write down a value of x, in the given interval, where $g(x + 5°) - f(x + 5°) = 1$.

1.5 Write down TWO values of x in the given interval, where $\frac{g(x)}{f^{/}(x)}$ is undefined.

1.6 Write down the value of p if $\sum_{x=0°}^{p} 2\cos x = 0$

2 In the diagram below, the graphs of $f(x) = a\sin 2x$ and $g(x) = \tan bx$ for $x \in [-90°; 90°]$ are drawn. P $(-68{,}53\ ; -0{,}68)$ and Q are the points of intersection of f and g.



2.1 Value of a

2.2 Coordinates of Q

2.3 x values of turning points of h, if $h(x) = f(x + 30°)$

2.4 Value(s) of x where $0{,}68 < g(x) \leq 1$

2.5 Value of m where $f(x + m) = -cos 2x$

2.6 Value of b

3. Given : $f(x) = \cos\frac{x}{2}$ and $g(x) = \sin(x - 30°)$ for $x \in [-180°; 180°]$.

The graphs f and g intersect at A and B

3.1 Use the grid provided to draw sketch graphs of f and g on the same set of axes. Show clearly all the intercepts on the axes and the coordinates of the turning points.

3.2 Calculate the x coordinates of A and B.

3.3 State the period of graph f.

3.4 Determine the values of $x \in [-90°; 180°]$ for which $g(x).f(x) < 0$.

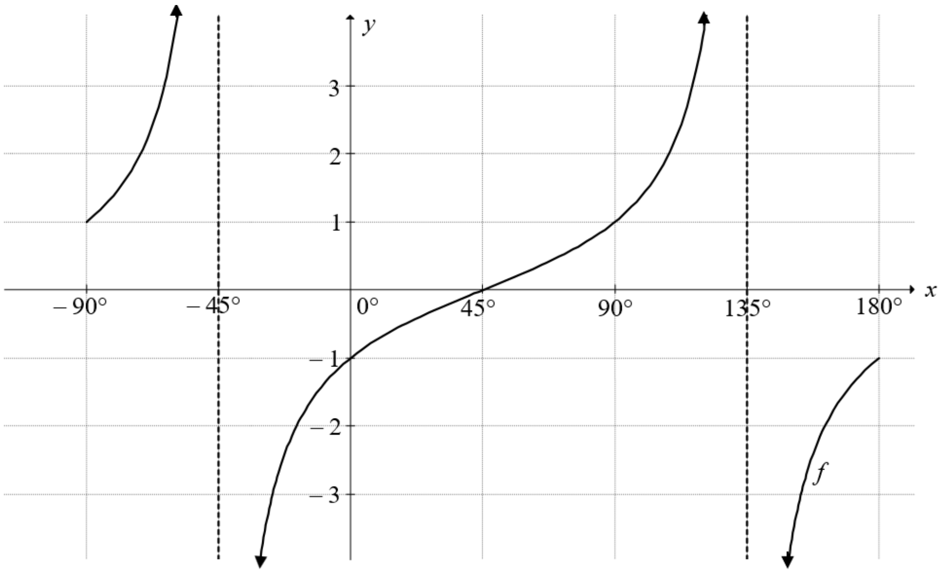
3.5 Given that the general solution of $f(x) = g(x)$ is: $x = 80° - k.240°, k \in z$. Determine the x values of point A and B.

3.6 For which value(s) of x will.

3.6.1 $f(x) > g(x)$

3.6.2 $f'(x) . g(x) > 0$ where $x > 0°$

4. In the diagram below, the graph of $f(x) = tan(x - 45°)$ is drawn for $x \in [-90°; 180°]$.



4.1 Write down the period of f.

4.2 Draw the graph of $g(x) = -cos\,2x$ for the interval $x \in [-90°; 180°]$ on the grid given in the ANSWER BOOK. Show ALL intercepts with the axes, as well as the minimum and maximum points of the graph

4.3 Write down the range of g.

4.4 The graph of g is shifted $45°$ to the left to form the graph of h. Determine the equation of h in its simplest form.

4.5.1 Use the graph(s) to determine the values of x in the interval $x \in [-90°; 90°]$ for which:

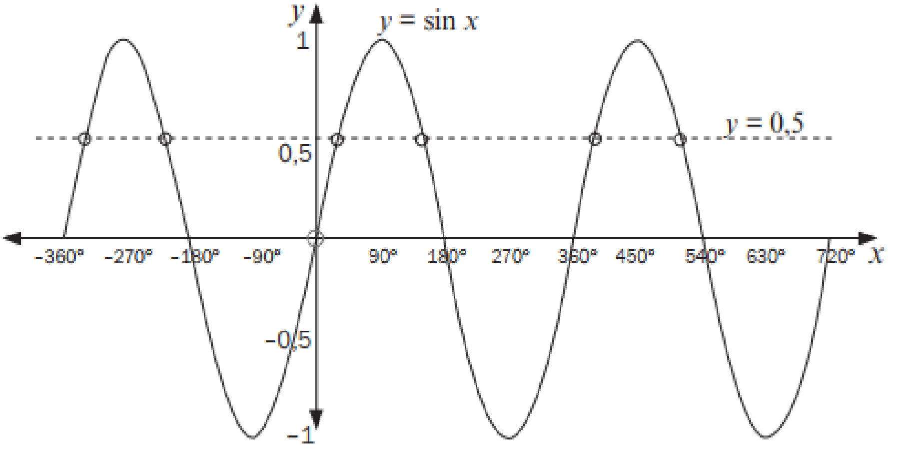
4.5.1 $f(x) > 1$

4.5.2 $2\,cos\,2x - 1 > 0$

SECTION 5: Trigonometric Equations

An equation with an unknown involving one or more trigonometric ratio of an unknown angle is called **trigonometric equation.** The solution of a trigonometric equation is the value of an unknown angle that satisfies the equation.

For example, $\sin\theta = \frac{1}{2}$, we know that θ could be 30°. However, there are other values for θ in other quadrants. Have a look at the graph for $\sin\theta = \frac{1}{2}, \theta \in [-360°; 720°]$ There are six values for θ between $-360°$ and 720°



Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature).

Basic	Squares	Co-functions	Factorizing
• Isolate the trig ratio • Determine the reference angle • Choose the quadrant	• Do ALL four quadrants	• Sin and cos with different angles • The angle you change is the reference angle.	• Use the principles of Algebra.

Worked example

Given the equation $sinx = -\frac{\sqrt{3}}{2}$, find the general solution.

a. Solving with the aid of the CAST DIAGRAM

- Find the inverse of $sinx$ on both sides, to find the reference angle.

  [We always use the positive values].

  $x = sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60°$

  Reference angle (RA) = 60°

- Use the CAST diagram to check the quadrants where the given function is negative.

  Sine is negative in 3rd and 4th quadrants

- Find the general solution on both quadrants:

❖ The general solution for 3rd quadrants is $x = 180° + RA + k.360°, k \in \mathbb{Z}$

$$x = 180° + 60° + k.360°$$
$$x = 240° + k.360°$$

or

❖ The general solution for 4th quadrant is $x = 360° - RA + k.360°, k \in \mathbb{Z}$

$$x = 360° - 60° + k.360°$$
$$x = 300° + k.360°$$

b. Solve the general solution using graphical method.

The solution for $sinx = -\frac{\sqrt{3}}{2}$

Is

$$x = sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + k.360°, k \in \mathbb{Z}$$
$$x = -60° + k.360°$$

or

$$x = 180° - sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + k.360° \qquad , k \in \mathbb{Z}$$
$$x = 240° + k.360°$$

Generally,

- if $sin\,\theta = a$, the general solution is given by:

  $\theta = sin^{-1}(a) + k.360°, k \in \mathbb{Z}$

  or

  $\theta = 180° - sin^{-1}(a) + k.360°, k \in \mathbb{Z}$

- If $cos\theta = a$, the general solution is given by:

  $\theta = \pm cos^{-1}(a) + k.360°, k \in \mathbb{Z}$

- If $\tan\theta = a$,the general solution is given by:

  $\theta = tan^{-1}(a) + k.180°, k \in \mathbb{Z}$

NB : Substitute "***a***" with its sign, if "***a***" is positive substitute as positive, if it is negative, substitute as negative

Worked example 1

Find the general solutions of the following equations:

a) $cos\,\theta = -0.25$

$b)$ $sin\,2\theta = 0.766$

$c)$ $cos\,(\theta - 15) = \frac{1}{2}$

$d)$ $3sin\,(\theta - 15) + 1 = -0.456$

Solution

a) $\boldsymbol{cos\,\theta = -0.25}$

$\theta = \pm cos^{-1}(-0.25)$

$\theta = \pm 104.48 + k.360, \quad k \in Z$

b) $\boldsymbol{sin\,2\theta = 0.766}$

$reference\ angle = sin^{-1}(0.766)$

$= 50$

$2\theta = 50 + k.360 \quad or \quad 2\theta = 180 - 50 + k.360, \quad \boldsymbol{k} \in \boldsymbol{Z}$

$\theta = 25 + k.180 \quad or \quad \theta = 65 + k.180$

c) $\boldsymbol{cos\,(\theta - 15) = \frac{1}{2}}$

$\theta = \pm cos^{-1}\left(\frac{1}{2}\right)$

$\theta - 15 = \pm 60 + k.360, \quad k \in Z$

$\theta = 15 + 60 + k.360 \quad or \quad \theta = 15 - 60 + k.360$

$\theta = 75 + k.360 \quad or \quad \theta = -45 + k.360$

$\boldsymbol{d})$ $\boldsymbol{3sin\,(\theta - 15) + 1 = -0.456}$

$\sin(\theta - 15) = -1.456$

$sin\,(\theta - 15) = -\frac{182}{375}$

$reference\ angle = sin^{-1}(-\frac{182}{375})$

$= -29.03$

$\theta - 15 = -29.03 + k.360 \quad or \quad \theta - 15 = 180 - (-29.03) + k.360, \boldsymbol{k} \in \boldsymbol{Z}$

$\theta = -14.03 + k.360 \quad or \quad \theta = 209.03 + k.360$

Worked example 4

(a) For which value(s) of x will $\frac{2tanx - sin2x}{2\sin^2 x}$ be undefined in the interval $0° \leq x \leq 180°$

(b) Prove the identity: $\frac{2tanx - sin2x}{2\sin^2 x} = tanx$

Solutions

a) The identity is undefined for

as: $2\sin^2 x = 0$

$\therefore \underline{\sin x = 0:}\quad x = 0°\ ;\ 180°$

or

$\underline{\tan x = \infty:}\quad x = 90°$

$\therefore x = 0°\ ;\ 90°\ ;\ 180°$

b) LHS

$$= \frac{2\tan x - \sin 2x}{2\sin^2 x}$$

$$= \frac{2\left(\frac{\sin x}{\cos x}\right) - 2\sin x\cos x}{2\sin^2 x}$$

$$= \left(\frac{2\sin x - 2\sin x\cos^2 x}{\cos x}\right) \times \frac{1}{2\sin^2 x}$$

$$= \frac{2\sin x(1-\cos^2 x)}{\cos x} \times \frac{1}{2\sin^2 x}$$

$$= \frac{2\sin x(\sin^2 x)}{\cos x} \times \frac{1}{2\sin^2 x}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

$$= \text{RHS}$$

Worked example 5

Determine the general solution of the following:

a. $6\ cosx - 5 = \frac{4}{cosx}$

b. $\sin 2x = cosx$

c. $3\sin x = 4cosx$

d. $4\cos^2 x + 4\sin x\cos x + 1 = 0$

e. $\cos 2x - 4\sin x + 5 = 0$

f. $\sin x + cosx = 1$

g. $\sin 2x + cosx = 0$

h. $2\cos x = \cos(60 - x)$

i. $3\,sin^2\,x - \sin 2x - \cos^2 \mathrm{x} =$

Solutions

a. $6cos^2x - 5cosx - 4 = 0$ [Similar to the quadratic equation $6k^2 - 5k - 4 = 0$]

$(2cosx + 1)(3cosx - 4) = 0$

$(2cosx + 1) = 0$ or $(3cosx - 4) = 0$

$2cosx = -1$ or $3cosx = 4$

$cosx = -\frac{1}{2}$ $cosx \neq \frac{4}{3}$

$\therefore cosx = -\frac{1}{2}$

General solutions

$$x = \pm cos^{-1}\left(-\frac{1}{2}\right) + k.360°, k \in \mathbb{Z}$$

$$x = \pm cos^{-1}\left(-\frac{1}{2}\right) + k.360°, k \in \mathbb{Z}$$

$x = cos^{-1}\left(-\frac{1}{2}\right) + k.360°$ or $x = -cos^{-1}\left(-\frac{1}{2}\right) + k.360°$

$x = 120° + k.360°$ or $x = -120° + k.360°$

b. $\sin 2x = cosx, x \in [-90°, 180°]$

$\cos(90° - 2x) = cosx$ [The co-function is used]

$90° - 2x = \pm cos^{-1}\ cos(x\ ) + k.360°, k \in \mathbb{Z}$

$90° - 2x = x + k.360°$ or $90° - 2x = -x + k.360°$

$-3x = -90° + k.360°$ $-x = -90° + k.360°$

$x = 30° - k.120°$ $x = 90° - k.360$

c. $3\sin x = 4\cos x$ [Divide both sides by $\cos x$ to create $\tan x$ on LHS]

$\frac{3\sin x}{\cos x} = \frac{4\cos x}{\cos x}$ [Trig identity for $\tan x$]

$3\tan x = 4$

$\tan x = \frac{3}{4}$

Ref angle $= 53{,}13°$

$x = 53{,}13° + k180°\ k \in \mathbb{Z}$ (3)

d. $4\cos^2 x + 4\sin x\cos x + 1 = 0$

$4\cos^2 x + 4\sin x\cos x + (\sin^2 x + \cos^2 x) = 0$

$\boxed{\sin^2 x + \cos^2 x = 1}$

$5\cos^2 x + 4\sin x\cos x + \sin^2 x = 0$

$(5\cos x + \sin x)(\cos x + \sin x) = 0$

$5\cos x + \sin x = 0$ or $\cos x + \sin x = 0$

$\frac{5\cos x}{\cos x} = \frac{-\sin x}{\cos x}$ or $\frac{\cos x}{\cos x} = \frac{-\sin x}{\cos x}$

$5 = -\tan x \therefore \tan x = -5$ | $1 = -\tan x \therefore \tan x = -1$

Reference angle $= 78{,}69°$ | Reference angle $= -45°$

$x = 180° - 78{,}69° + k180°$ or $\therefore x = 180° - 45° + k180°$

$\therefore x = 101{,}3° + k180°$ | $\therefore x = 135° + k180°\quad k \in \mathbb{Z}$

e. $\cos 2x - 4\sin x + 5 = 0$

$1 - 2sin^2 x - 4\sin x + 5 = 0$ [$\cos 2x = 1 - 2sin^2 x$]

$-2sin^2 x - 4\sin x + 6 = 0$

$sin^2 x - 2\sin x - 3 = 0$ [Divided by -2]

$(\sin x - 3)(\sin x + 1) = 0$ [Factorize]

$\sin x \neq 3$ or $\sin x = -1$

$x = 90 + k.360,\ k \in Z$

f. $\sin x + cosx = 1$

$(\sin x + cosx)^2 = 1^2$

$sin^2 x + 2\,\mathrm{sin}x.\mathrm{cosx} + cos^2 x = 1$

$sin^2 x + cos^2 x + 2\,\mathrm{sin}x.\mathrm{cos}x = 1$ [$sin^2 x + cos^2 x = 1$]

$1 + 2\,\mathrm{sin}x.\mathrm{cos}x = 1$

$\mathrm{sin}2x = 0$ [$2\,\mathrm{sin}x.\mathrm{cos}x = \mathrm{sin}2x$]

$2x = 90° + k.360,\ k \in Z$

$x = 45° + k.180$ [divide EVERY TERM by 2]

g. $\sin 2x + \sin x = 0$

$2 \sin x \cos x + \sin x = 0$ [Change double angles to single angles: $\sin 2x = 2\sin x.\cos x$]

$\sin x\,(2\cos x + 1) = 0$ [Factorise by taking out the common factor of $\sin x$]

$2\cos x + 1 = 0$ or $\sin x = 0$

$\cos x = -\frac{1}{2}$ $x = 0° + n.180°$

$x = \pm 120° + k.360°,\ k \in \mathbb{Z}$

h. $2 \cos x = \cos(60° - x)$

$2\,cos x = \cos 60°.cos x + sin 60°.sin x$

$2 = \cos 60° + \frac{\sin 60°.\sin x}{\cos x}$ [Divide everything by $\cos x$]

$2 = \frac{1}{2} + \frac{\sqrt{3}}{2}.\tan x$

$\frac{3}{2} = \frac{\sqrt{3}}{2}.\tan x$

$\frac{3}{\sqrt{3}} = tan x$

Reference angle $= 60°$

$x = 60 + k.180°,$ $k \in \mathbb{Z}$

ACTIVITIES

1. Determine the general solution for the following:
   - 1.1 $\sin^2 x - \sin x = \cos^2 x$
   - 1.2 $\sin^2 x + \sin x \cos x = 0$
2. Determine the general solution of : $3cos2x = 1 + 5cos x$
3. Solve for x: $4\sin^2 x - 3sin x - 1 = 0$ for $x \in [0°; 360°]$
4. Determine the general solution of $tan\, x = 2\, sin\, 2\,x$ where $\cos x < 0$.
5. Calculate the value of x, if $x \in [-180°; 360°]$ $cos2x = cosx + 2$

BIBLIOGRAPHY

1	**CAPS Document**
2	**Examination guidelines**
3	**Dbe November 2008 – 2024 Past Papers**
4	**Dbe June and Feb/Mar 2009 – 2024 Past Papers**
5	**Mind The Gap Textbook**
6	**Mind Action Series Textbook**
7	**JENN 2022 – 2024 Manuals**