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MATHEMATICS

GRADE 12

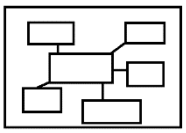
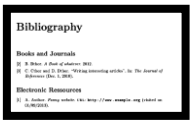
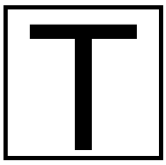
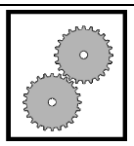
SPRING SCHOOL

TEACHER AND LEARNER CONTENT MANUAL

2026

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ICON DESCRIPTION

 MIND MAP	 EXAMINATION GUIDELINE	 CONTENTS	 ACTIVITIES
 BIBLIOGRAPHY	 TERMINOLOGY	 WORKED EXAMPLES	 STEPS

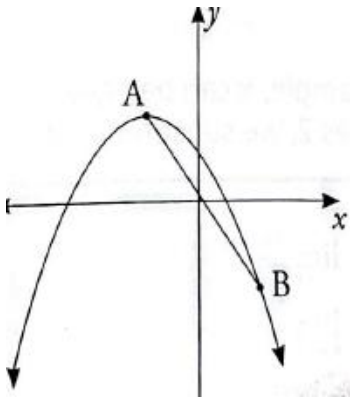
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SECTION 1 : CALCULUS

1.1 AVERAGE GRADIENT

The average gradient of a function on a curve is the gradient of the straight line passing through these two points. It can be determined by $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$, where $(x_1 ; f(x_1)$ and $(x_2 ; f(x_2)$ are the two points we want average gradient of .



Therefore, the average gradient will be: $m_{ave} = \frac{f(b) - f(a)}{b - a}$

WORKED EXAMPLE 1

Find the average gradient of $f(x) = x^2 - 16$ between $x = 5$ and $x = 9$.

Solution:

$$m_{ave} = \frac{f(b) - f(a)}{b - a}$$
$$= \frac{(9)^2 - 16 - ((5)^2 - 16)}{9 - 5}$$
$$= \frac{65 - 9}{4}$$
$$= 14$$

SECTION 1.2 : FIRST PRINCIPLES

The question WILL ask you to differentiate using FIRST PRINCIPLES (definition), you must use the formula the following equation : $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

Practice various examples of the following functions : 1. $f(x) = ax + k$

2. $f(x) = ax^2 + bx + c$
3. $f(x) = ax^3$
4. $f(x) = \frac{a}{x}$

The constants a, b and c can be any constant variable or whole numbers or fractions or any type of numbers.

Differentiating from the First Principles (from the Definition)

To differentiate from the first principles, we recommend you follow the following steps:

Step 1: Copy the given function $f(x)$

Step 2: Simplify $f(x+h)$

Step 3: Simplify $f(x+h)-f(x)$

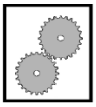
**** Always make sure that $f(x)$ is in brackets*

Step 4: Copy the **CORRECT** formula from the formula sheet; $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Step 5: Substitute answer from ***step 3*** in ***step 4***

Step 6: Simplify ***step 5***

Step 7: **Write your ANSWER with CONFIDENCE**.



WORKED EXAMPLE 2

Find, from first principles, the gradient of $f(x)=2x$ at any point.

$f(x)=2x$

$f(x+h)=2(x+h)$

$=2x+2h$

$f(x+h)-f(x)=2x+2h-(2x)$

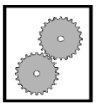
$=2h$

$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$f'(x)=\lim_{h\to 0}\frac{2h}{h}$

$f'(x)=\lim_{h\to 0}2$

$f'(x)=2$



WORKED EXAMPLE 3

Find, from first principles, the derivative of $f(x) = 6$ at any point.

$f(x) = 6$

$f(x+h) = 6$ (*since there is no x in the given function it therefore means* $f(x+h) = f(x)$)

$f(x+h) - f(x) = 6 - 6$

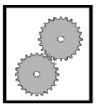
$= 0$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{0}{h}$

$f'(x) = \lim_{h\to 0} 0$

$f'(x) = 0$



WORKED EXAMPLE 4

Find $f'(x)$ from first principles if $f(x) = 3x^2 + x$

$f(x) = 3x^2 + x$

$f(x+h) = 3(x+h)^2 + (x+h)$

$= 3(x^2 + 2xh + h^2) + x + h$

$= 3x^2 + 6xh + 3h^2 + x + h$

$f(x+h) - f(x) = 3x^2 + 6xh + 3h^2 + x + h - (3x^2 + x)$

$= 3x^2 + 6xh + 3h^2 + x + h - 3x^2 - x$

$= 6xh + 3h^2 + h$

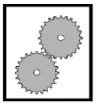
$= h(6x + 3h + 1)$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{h(6x+3h+1)}{h}$

$f'(x) = \lim_{h\to 0} (6x + 3h + 1)$

$f'(x) = 6x + 1$



WORKED EXAMPLE 5

Find $f'(x)$ from first principles if $f(x) = -x^3$

$f(x) = -x^3$

$f(x+h) = -(x+h)^3$

$= -(x^2 + 2xh + h^2)(x+h)$

$= -(x^3 + 3x^2h + 3xh^2 + h^3)$

$= -x^3 - 3x^2h - 3xh^2 - h^3$

$f(x+h) - f(x) = -x^3 - 3x^2h - 3xh^2 - h^3 - (-x^3)$

$= -x^3 - 3x^2h - 3xh^2 - h^3 + x^3$

$= -3x^2h - 3xh^2 - h^3$

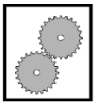
$= h(-3x^2 - 3xh - h^2)$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{h(-3x^2-3xh-h^2)}{h}$

$f'(x) = \lim_{h\to 0} (-3x^2 - 3xh - h^2)$

$f'(x) = -3x^2$



WORKED EXAMPLE 6

Determine, from the first principles, the derivative of $f(x) = \frac{1}{x}$.

$f(x) = \frac{1}{x}$

$f(x+h) = \frac{1}{x+h}$

$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x}$

$= \frac{1(x)-1(x+h)}{(x+h)x}$

$= \frac{x-x-h}{x^2+xh}$

$= \frac{-h}{x^2+xh}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{\left(\frac{-h}{x^2+xh}\right)}{h}$

$f'(x) = \lim_{h\to 0} \left(\frac{-h}{x^2+xh} \times \frac{1}{h}\right)$

$f'(x) = \lim_{h\to 0} \frac{-1}{x^2+xh}$

$f'(x) = \frac{-1}{x^2}$

SECTION 1.3 : Rules of Differentiation

We will now determine the derivative using the rules of differentiation. The following notations are used for the derivative:

$\frac{dy}{dx}$ if y = ... is given

$f'(x)$ if $f(x)$ is given

All the above notations say we need to differentiate with respect to x.

Necessary prior knowledge:

- How to multiply brackets
- How change surd form to exponential form: $\sqrt[n]{x^m} = x^{\frac{m}{n}}$
- How to remove a variable from being a denominator: $\frac{1}{a^n} = a^{-n}$

If $f(x) = ax^n$, then the derivative will be $f'(x) = anx^{n-1}$ **(Power rule)**

NB: 1. You may not differentiate when your expression has **brackets** – **multiply it out first.**

2. You may not differentiate when x **is the denominator**- write $\frac{1}{x^n}$ as x^{-n}

3. You may not differentiate when there is a surd write $\sqrt{x}$ as $x^{\frac{1}{2}}$

***Rule 1*:** If $f(x) = ax^n$, then $f'(x) = a.nx^{n-1}$

Example

$f(x) = -4x^3$, then $f'(x) = -12x^2$

***Rule 2*:** If $f(x) = ax$, then $f'(x) = a$

Example

$f(x) = -4x$, then $f'(x) = -4$

***Rule 3*:** If $f(x) = a$, then $f'(x) = 0$

Example

$f(x) = 9$, then $f'(x) = 0$

***Rule 4*:** If $f(x) =$ letter (not x) which is a treated as a constant, then $f'(x) = 0$

Example

If $f(x) = k$, then $f'(x) = 0$

Rule 5: $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$

Example

Determine $\frac{dy}{dx}$ if $y = -4x^2 + 7x - 8$, then $\frac{dy}{dx} = -8x + 7$

Example (Using the symbols $f'(x)$ and $\frac{dy}{dx}$)

a) If $f(x) = x^4 + 6x + 7$, determine $f'(x)$, the derivative of f.

Rule 1 **Rule 2** **Rule 3**

$f'(x) = 4x^{4-1} + 6 + 0$

$\therefore f'(x) = 4x^3 + 6$

b) If $g(x) = 4x^5 + 5\pi + t^2$, determine $g'(x)$, the gradient of g.

Rule 1 **Rule 3** **Rule 4**

$g'(x) = 4 \times 5x^{5-1} + 0 + 0$

$\therefore g'(x) = 20x^4$

c) If $y = \frac{1}{3}x^{-6} + x + \frac{x^3}{3}$, determine $\frac{dy}{dx}$, the derivative of y with respect to x.

$y = \frac{1}{3}x^{-6} + x + \frac{x^3}{3}$

Rewrite $\frac{x^3}{3}$ as $\frac{1}{3}x^3$ clearly separating the coefficient and our respected variable

$\therefore y = \frac{1}{3}x^{-6} + x + \frac{1}{3}x^3$

Now use the rules to differentiate (find the derivative)

$\frac{dy}{dx} = \frac{1}{3} \times -6x^{-6-1} + 1 + \frac{1}{3} \times 3x^{3-1}$ (Rule 1, Rule 2, Rule 1)

$\therefore \frac{dy}{dx} = -2x^{-7} + 1 + x^2$

Note: The symbols $f'(x)$ and $\frac{dy}{dx}$ refer to the gradient or derivative of functions written in the form $f(x) =$ … and $y =$..

Example (Using the symbol D_x)

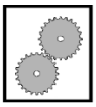
Determine $D_x\ [\pi x^2 + \pi^2 + x^{-2}]$

$D_x\ [\pi x^2 + \pi^2 + x^{-2}]$

$= \pi \times 2x^{2-1} + 0 + (-2)x^{-2-1}$

$= 2\pi x - 2x^{-3}$

Note: Be aware of the placement of the equal signs and the use of the symbols in the previous examples.



WORKED EXAMPLE 6

Determine the following:

a) $f'(x)$ if $f(x) = (4x - 3)^2$

b) $\frac{dy}{dx}$ if $y = \frac{3}{\sqrt{x}}$

c) $D_x \left[\frac{x^2-x-6}{2x^2}\right]$

d) $D_x \left[\frac{3x^3-7x^2-6x}{3-x}\right]$

Solution

a) $f'(x)$ if $f(x) = (4x - 3)^2$

Here we first need to expand and simplify so as to remove the brackets.

$f(x) = (4x - 3)^2$

$f(x) = 16x^2 - 24x + 9$

$f'(x) = 16 \times 2x^{2-1} - 24 + 0$

$\therefore f'(x) = 32x - 24$

b) $\frac{dy}{dx}$ if $y = \frac{3}{\sqrt{x}}$

Here we need to first get rid of the surd.

$y = \frac{3}{x^{\frac{1}{2}}}$

We now need to take note of our respected variable which is on the denominator.

$y = 3x^{-\frac{1}{2}}$

$\therefore \quad \frac{dy}{dx} = -\frac{1}{6}x^{-\frac{3}{2}}$

c) $D_x \left[\frac{x^2-x-6}{2x^2}\right]$ with 1 term on the denominator; distribute it as shown below.

$= D_x \left[\frac{x^2}{2x^2} - \frac{x}{2x^2} - \frac{6}{2x^2}\right]$

$= D_x \left[\frac{1}{2} - \frac{1}{2x} - \frac{3}{x^2}\right]$

$= D_x \left[\frac{1}{2} - \frac{1}{2}x^{-1} - 3x^{-2}\right]$

$= 0 - \frac{1}{2}(-1)x^{-1-1} - 3(-2)x^{-2-1}$

$= \frac{1}{2}x^{-2} + 6x^{-3}$

d) $D_x \left[\frac{3x^3-7x^2-6x}{3-x}\right]$ with more than 1 term on the denominator; there is need to factorise the numerator.

$= D_x \left[\frac{x(3x^2-7x-6)}{3-x}\right]$

$= D_x \left[\frac{x(3x+2)(x-3)}{3-x}\right]$

$= D_x \left[\frac{x(3x+2)(x-3)}{-(x-3)}\right]$

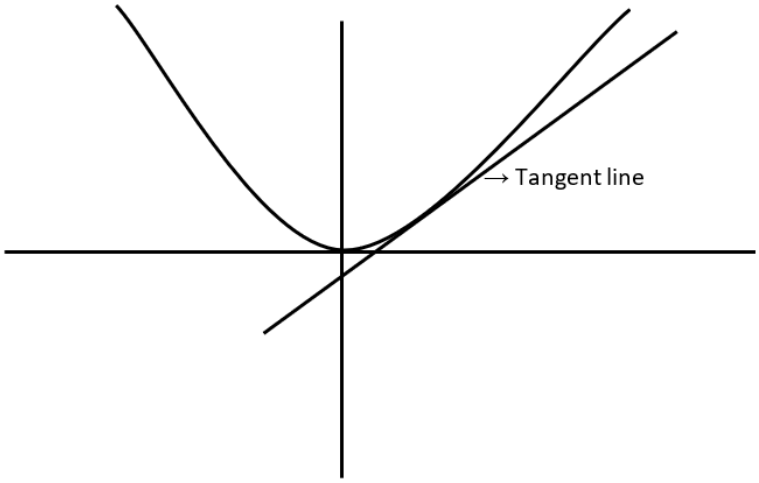
$= D_x \left[-x(3x+2)\right]$

$= D_x \left[-3x^2 - 2x\right]$

$= -6x - 2$

SECTION 1.4 : The Equations of a Tangent to The Curve

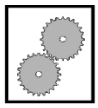
A tangent is a straight line that touches a curve at one point. A central principle of calculus is that at any point $x = a$ on the graph of a function f, the gradient of f at $x = a$ is the same as the gradient of the tangent to f at $x = a$. In other words, $m_t = f'(a)$.



How to determine the equation of a tangent.

Given the function $f(x)$ and the x – coordinate of the point of tangency. Do the following steps :

1. Differentiate the function given, $f'(x)$, then substitute by the given x to get the gradient.
2. Substitute the given x value into original equation to obtain the y-coordinate of the point of tangency $(x_1 ; y_1)$.
3. Substitute the obtained m and $(x_1 ; y_1)$ into $y - y_1 = m(x - x_1)$ and simplify.



WORKED EXAMPLE 7

Determine the equation of a tangent to $f(x) = x^2 + x$ at the point (1; 2)

$f'(x) = 2x + 1$

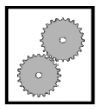
$m = f'(1) = 2(1) + 1$

$m = 3$

$y - y_1 = m(x - x_1)$

$y - 2 = 3(x - 1)$

$\therefore \quad y = 3x - 1$



WORKED EXAMPLE 8

The graph of $g(x) = -3x^3 + 2x^2 - 3x + 5$ is given.

Determine the equation of the tangent to the graph of g at the point where $x = -1$.

$g(-1) = -3(-1)^3 + 2(-1)^2 - 3(-1) + 5$

$= 13 \qquad\qquad (-1; 13)$

$g(x) = -3x^3 + 2x^2 - 3x + 5$

$\therefore g'(x) = -9x^2 + 4x - 3$

$m = g'(-1) = -9(-1)^2 + 4(-1) - 3$

$m = -16$

Therefore, the equation of the tangent is:

$y - y_T = m_t(x - x_T)$

$y - 13 = -16(x - (-1))$

$y - 13 = -16(x + 1)$

$y - 13 = -16x - 16$

$y = -16x - 3$

How to determine the coordinates of the point of tangency (point of contact).

Given the function $f(x)$ and the gradient. Do the following 2 substitutions:

1. Substitute the given gradient m into the derivative $f\,'(x)$ to find the x coordinate.

2. Substitute the obtained x coordinate into $f(x)$ to obtain the y – coordinate of the point of tangency $(x_1\ ;\ y_1)$.

Example

Determine the point of contact between the tangent with the gradient of 10 and the curve $f(x) = x^2 + 2x - 1$.

Solution:

$f\,'(x) = 2x + 2$

$10 = 2x + 2$

$8 = 2x$

$4 = x$

$f(4) = (4)^2 + 2(4) - 1$

$= 23$

$\therefore$ The point of contact is $(4\ ;\ 23)$

SECTION 1.5 : CUBIC FUNCTION

SKETCHING CUBIC FUNCTION

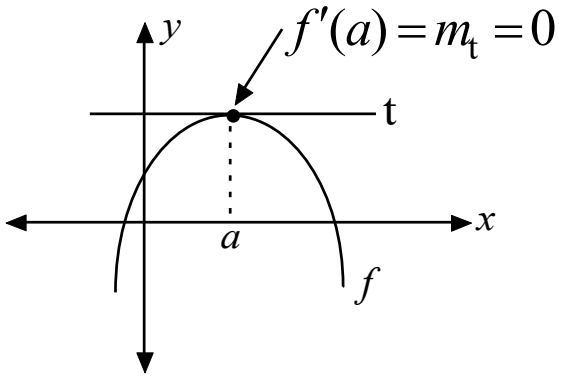
You must be able to sketch the cubic function, giving the special attention to the following key features:

- Intercepts (both x and y intercepts)
- Turning points (stationary points)

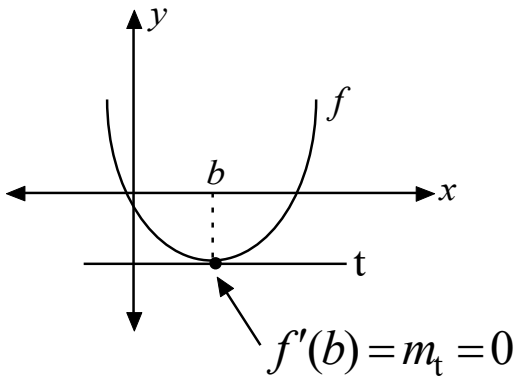
Clearly indicating the intercepts with the axis as well as the turning points, every time you are asked to sketch a cubic function.

Coordinates of Turning Points (Stationary Points)

At any ***stationary point*** on a graph *f*, the gradient is zero. If $x = a$ is a stationary point, then $f'(a) = 0$ since the gradient of the tangent line at $x = a$ is zero (the line is horizontal). At a stationary point, the function neither increases nor decreases.



At $x = a$ we have a local ***maximum***



At $x = b$ we have a local ***minimum***.

Given the function $f(x)$ and then required to get the co-ordinates of the turning point, we recommend the following steps:

Step 1: Determine $f'(x)$

Step 2: Equate $f'(x)$ to 0, $f'^{(x)} = 0$

Step 3: Solve for x

Step 4: Substitute the values of x (obtained in *Step 3*) in the original function, $f(x)$, to get the corresponding $y-$values.

Step 5: You now mastered it; you have to write the coordinates of the turning points.

Points of Inflection (Concavity)

Point of inflection

Method 1

- Equating the second derivative to zero. $f''(x) = 0$

1. Equate the second derivative to zero and solve for x. You now have x-value of point of inflection.
2. Substitute the x-value into the original function, e.g $f(x)$, to get the corresponding y-value, and thus the point of inflection.

Method 2

- Using coordinates of the turning points. The point of inflection lies half-way between the turning points of a cubic function
- Find the coordinates of the point of inflection by finding the midpoint of the turning points.

***Concave up*: $f''(x) > 0$**

***Concave down*: $f''(x) < 0$**

Example

Consider the cubic function $h(x) = 2x^3 + 4x^2 - 1$.

a) Determine the coordinates of the turning points.

b) Determine the coordinates of the point of inflection

Solution

a) $h(x) = 2x^3 + 4x^2 - 1$

$h'(x) = 6x^2 + 8x$

$6x^2 + 8x = 0$

$2x(3x + 4) = 0$

$2x = 0$ or $3x + 4 = 0$

$x = 0$ or $x = -\frac{4}{3}$

$h(0) = 2(0)^3 + 4(0)^2 - 1$ or $h(-\frac{4}{3}) = 2\left(-\frac{4}{3}\right)^3 + 4\left(-\frac{4}{3}\right)^2 - 1$

$h(0) = -1$ or $h(0) = \frac{37}{27}$

Therefore; coordinates of turning points are:

$(0; -1)$ and $\left(-\frac{4}{3}; \frac{37}{27}\right)$

b) Coordinates of the point of inflection

$= \left(\frac{0+\left(-\frac{4}{3}\right)}{2}; \frac{-1+\frac{37}{27}}{2}\right)$

$= \left(-\frac{2}{3}; \frac{5}{27}\right)$

Factorizing a Cubic Function

Different methods can be employed. Regardless of the method used to get all factors, it is worth noting that the factor theorem plays a significant role to at least get the first factor.

The factors theorem states that if $f(a) = 0$; if follows that $x - a$ is a factor of $f(x)$.

Examination question can either request only the factorisation of cubic function or to determine the coordinates of the x −intercepts.

Example

Given: $f(x) = x^3 - 8x^2 + 5x + 14$.

a) Factorise $f(x)$ completely.

b) What are coordinates of the turning points?

c) For which values of x is $f(x)$ concave down?

Solution

a) $f(x) = x^3 - 8x^2 + 5x + 14$

Find a number such that if it is substituted in $f(x)$ we obtain 0

$f(2) = (2)^3 - 8(2)^2 + 5(2) + 14$

$f(2) = 0$

$\therefore (x - 2)$ is a factor

$f(x) = (x - 2)(x^2 + bx - 7)$

Hence;

$bx^2 - 2x^2 \equiv -8x^2$

$b - 2 = -8$

$b = -6$

$f(x) = (x - 2)(x^2 - 6x - 7)$

$f(x) = (x - 2)(x - 7)(x + 1)$

b) $f(x) = x^3 - 8x^2 + 5x + 14$

$f'(x) = 3x^2 - 16x + 5$

$3x^2 - 16x + 5 = 0$

$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(3)(5)}}{2(3)}$

$x = \frac{1}{3}$ or $x = 5$

$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - 8\left(\frac{1}{3}\right)^2 + 5\left(\frac{1}{3}\right) + 14$ or $f(5) = (5)^3 - 8(5)^2 + 5(5) + 14$

$f\left(\frac{1}{3}\right) = \frac{400}{27}$ or $f(5) = -36$

Therefore; coordinates of turning points are:

$\left(\frac{1}{3}; \frac{400}{27}\right)$ and $(5; -36)$

c) $f''(x) < 0$

$6x - 16 < 0$

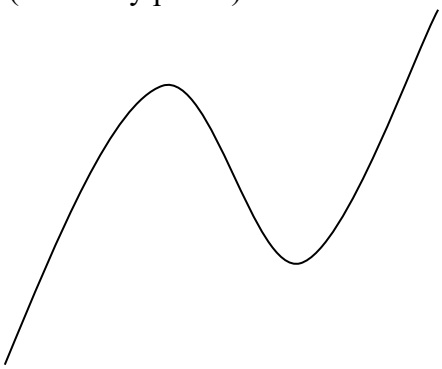
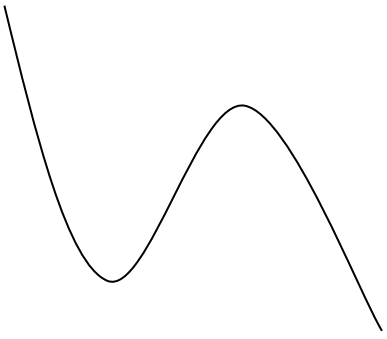
$6x < 16$

$x < \frac{8}{3}$

Sketching the Cubic Function

The standard form of a cubic function is given by: $y = ax^3 + bx^2 + cx + d$

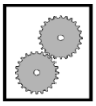
Basic shapes of a cubic function.

Shape if $a > 0$	**Shape if $a < 0$**
Local maximum and minimum turning points (stationary points). 	Local maximum and minimum turning points (stationary points). 

Steps to follow when sketching:

N.B sometimes, if not most of the time, you will be directed by the question as to where to start

1) Determine the x - and y - intercepts
   a) For x intercept:
      i) Let y = 0
      ii) Find the factors and solve for x. (Factor theorem is important).
      iii) Calculate the corresponding y values.
   b) For y intercept: Let $x = 0$ and solve for y
2) Determine the turning points
3) Sketch



WORKED EXAMPLE 10

Given the function $f(x) = -2x^3 + 12x^2 - 18x + 8$:

1. Determine the y intercept of f.
2. Show that $x - 1$ is a factor of f.
3. Hence, determine the $x - int$ of f.
4. Determine the coordinates of the turning points.
5. Sketch the graph of f, clearly indicate the intercepts with the axis as well as the turning points.

Solutions

1. $f(0) = -2(0)^3 + 12(0)^2 - 18(0) + 8$

   $= 8$ $\quad (0\,;8)$

2. $f(1) = -2(1)^3 + 12(1)^2 - 18(1) + 8$

   $= 0$ $\quad (1\,;0)$

3. $0 = (x-1)(-2x^2 + 10x - 8)$

   $x - 1 = 0$ or $-2x^2 + 10x - 8 = 0$

   $x = 1$ or $x^2 - 5x + 4 = 0$

   $x = 1$ or $(x-4)(x-1) = 0$

   $x = 1$ or $x = 4$ or $x = 1$

   $(1\,;0)$ and $(4:0)$

4. $f'(x) = -6x^2 + 24x - 18$

$0 = -6x^2 + 24x - 18$

$0 = x^2 - 4x + 3$

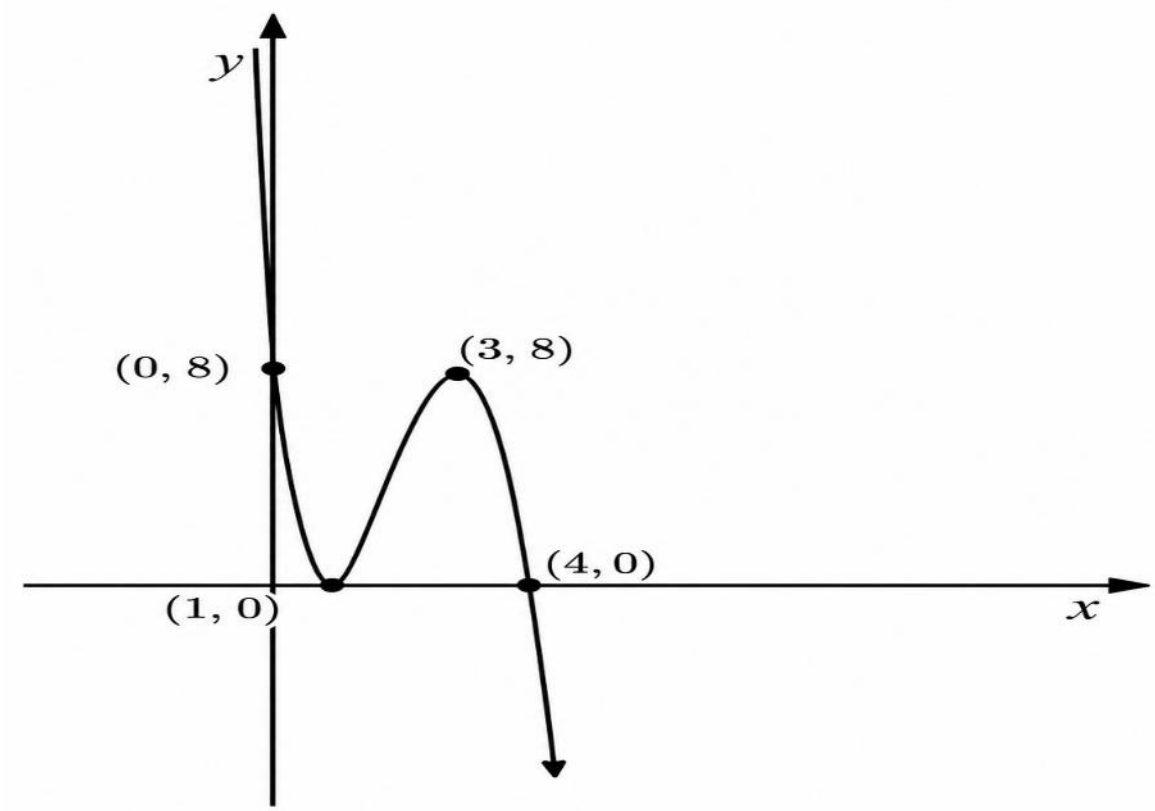
$0 = (x - 3)(x - 1)$

$x = 3$ or $x = 1$

$f(3) = 8$ and $f(1) = 0$

$(3; 8)$ and $(1; 0)$

5.



FINDING THE EQUATION OF CUBIC FUNCTION

It is important to note that the number of conditions given in the questions must be at least equal to the constants that need to be determined to formulate some equations. Conditions are usually given in the form of coordinates which can either be intercepts, turning points or any other points on the graph of the functions.

Forms of the cubic function:

Standard form: $f(x) = ax^3 + bx^2 + cx + d$

3 intercepts form: $f(x) = a(x - x_1)(x - x_2)(x - x_3)$

Where; x_1, x_2 and x_3 are x −intercept

2 intercepts form: $f(x) = a(x - x_1)(x - x_2)^2$

Where; x_1 is only an x −intercept and x_2 is both an x −intercept and a turning point.

The turning point and the point of contact of tangents provides more than 1 condition. With the turning point or the point of contact of a graph and a tangent, we can easily construct more than 1 equations.

If given coordinates of a turning point $f(x)$ (2 conditions):

i. The pair of coordinates can be substituted in $f(x)$

ii. The x −coordinate can be substituted in $f'(x)$ then equate to zero

If given coordinates of a point of contact of $f(x)$ and its tangent (2 conditions):

i. The pair of coordinates can be substituted in $f(x)$

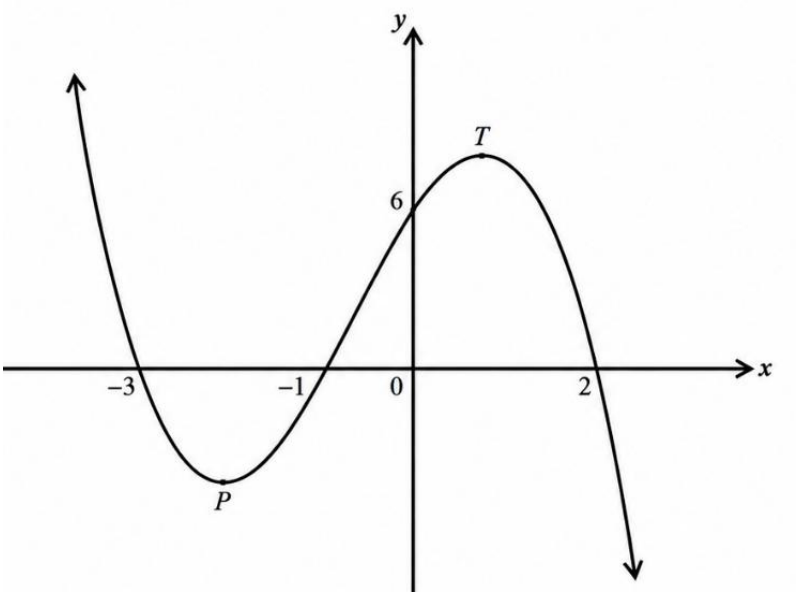
ii. The x −coordinate can be substituted in $f'(x)$ then equate to gradient of tangent at the point of contact.

If given any other coordinates on $f(x)$

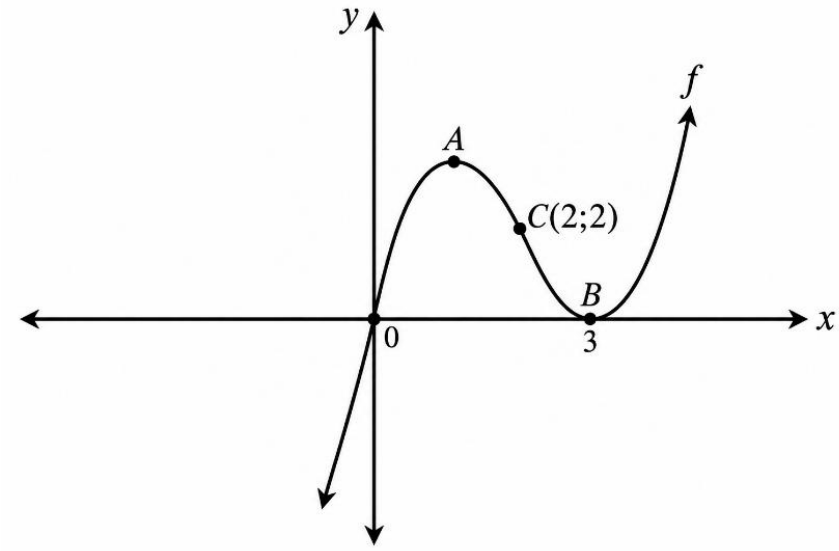
i. The pair of coordinates can be substituted in $f(x)$

WORKED EXAMPLE 11

1. The sketch below represents the graph of f defined by $f(x) = -x^3 + ax^2 + bx + c$. P and T are the turning points of f. The x-intercepts of the graph are $-3, -1$ and 2. The graph has a y-cut at 6. Determine the values of a, b and c.



2. The sketch below represents the graph of f defined be $f(x) = ax^3 + bx^2 + c$. A and B are the turning points of f. B (3;0) is also the x – intercept of f. C is a point with coordinates (2;2). The graph passes through the origin.



Solutions

1.

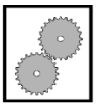
$$y = -(x+3)(x+1)(x-2)$$
$$y = -(x^3 + 2x^2 - 5x - 6)$$
$$= -x^3 - 2x^2 + 5x + 6$$
$$a = -2;\ b = 5;\ c = 6$$

2.

$$y = a(x - x_1)(x - x_2)^2$$
$$2 = a(2-0)(2-3)^2$$
$$2 = a(2)(1)^2$$
$$a = 1$$
$$y = 1(x)(x-3)^2$$
$$f(x) = x^3 - 6x^2 + 9x$$

Relating the Key Features of f, f' and f'' to Each Other

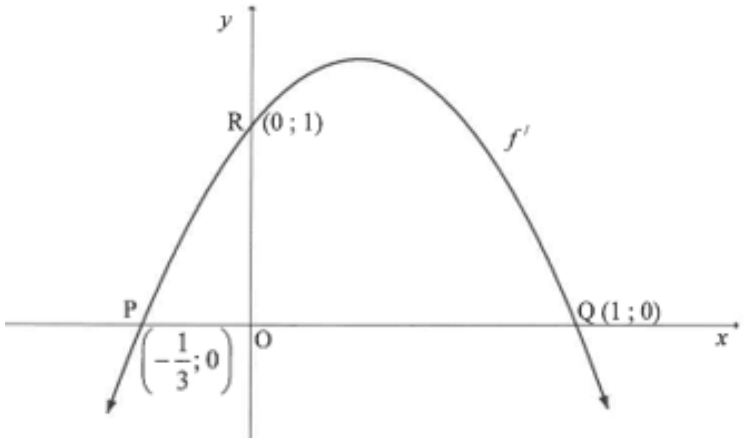
1. The value of x −intercept of f'' is the same as the x-value of the coordinates of the turning point of f'. It is also the x-value of the point of inflection of f.
2. The value of x-intercepts of f' are the same as x-values of the coordinates of the turning point of $\boldsymbol{f}$.



WORKED EXAMPLE 12

The graph of $y = f'(x) = mx^2 + nx + k$ is drawn below.

The graph passes the points $P\left(-\frac{1}{3}; 0\right), Q(1;0) and\ R(0;1)$. Determine the values of $m, n\ and\ k$.



Solution

$$f'(x) = mx^2 + nx + k$$

$$f'(x) = m\left(x + \frac{1}{3}\right)(x-1)$$

$$1 = m\left(0 + \frac{1}{3}\right)(0-1)$$

$$1 = -\frac{1}{3}m$$

$$\therefore m = -3$$

$$f'(x) = -3\left(x + \frac{1}{3}\right)(x-1)$$

$$f'(x) = -3\left(x^2 - \frac{2}{3}x - \frac{1}{3}\right)$$

$$f'(x) = -3x^2 + 2x + 1$$

$$\therefore n = 2$$

$$\therefore k = 1$$

ACTIVITIES

SECTION 1: CALCULUS

ACTIVITY 1

Determine $f'(x)$, from the first principles :

1. $f(x) = 5$
2. $f(x) = mx + c$
3. $f(x) = -x^2 - x$
4. $f(x) = -\frac{1}{2}x^2 - 3x$
5. $f(x) = px^2 - qx + r$
6. $f(x) = \frac{1}{2}x^2 - 3x + 6$
7. $f(x) = 2x^3$
8. $f(x) = -\frac{2}{x}$

ACTIVITY 2

Determine :

2.1 $\frac{d}{dx}(\pi^2)$

2.2 $D_x\left(x^4 - \sqrt[3]{x}\right)$

2.3 $\frac{dy}{dx}$ if $y = \frac{x^5+2}{x^2}$

ACTIVITY 3

Determine EACH of the following:

3.1 $\frac{dA}{dr}$ if $A = \pi r^2$

3.2 $D_x\left[\left(x - \sqrt{x}\right)^2\right]$

ACTIVITY 4

Determine:

4.1 $\frac{dy}{dx}$ if $y = \frac{2x^2-x-6}{2x+3}$

4.2 $f'(x)$ if $f(x) = \frac{3}{x^4} - \frac{x}{\sqrt{x}}$

4.3 $\frac{dy}{dx}$ if $y = a^2x + 6\sqrt{x}$

4.4 $g'(x)$ if $g(x) = (x+7)^3$

4.5 $\frac{dq}{dp}$ if $9p^2 - 3pq - q = 1$

ACTIVITY 5

Determine:	
5.1	$D_x[x^3 + 3x^2 - 6x + 5]$
5.2	$m'(x)$ if $m(x) = \frac{1}{x} + \frac{3x}{4}$
5.3	$f'(x)$ if $f(x) = \frac{x^3-27}{x^2-3x}$
5.4	$\frac{dx}{dy}$ if : $\sqrt[3]{y}.x = 2y^2 + 3$

ACTIVITY 6

6.1 Find the equation of the tangent to the following curves at a given value of x.

6.1.1 $y = x^2 - x - 6$ at $x = 1$

6.1.2 $y = -\frac{6}{x}$ at $x = -2$

6.2 The tangent to the curve of $g(x) = 2x^3 + px^2 + qx - 7$ at $x = 1$ has the equation $y = 5x - 8$.

6.2.1 Show that (1; -3) is the point of contact of the tangent to the graph of g.

6.2.2 Hence, calculate the values of p and q.

6.3 For which value of x is the tangent to the curve $y = x^2 + 3x - 5$ parallel to the line $3y + 2x = 6$?

ACTIVITY 7

1. Sketch the graph of $f(x) = x^3 - 12x^2 + 36x$. Show clearly all the intercepts with the axes as well as the coordinates of the stationery points.

ACTIVITY 8

Given: $f(x) = x^3 + 4x^2 - 7x - 10$

8.1 Write down the y - intercept of f.

8.2 Show that 2 is a root of the equation $f(x) = 0$.

8.3 Hence, factorise $f(x)$ completely.

8.4 If it is further given that the coordinates of the turning points are approximately at $(0{,}7\ ;\ -12{,}6)$ and $(-3{,}4\ ;\ 20{,}8)$, draw a sketch graph of f and label all intercepts and turning points.

8.5 Use your graph to determine the values of x for which $f'(x) < 0$

ACTIVITY 9

Sketch the graph of $f(x) = ax^3 + bx^2 + cx + d$ if:

- $f'(-3) = f'(5) = 0$
- $f(-3) = 5$ and $f(5) = -3$
- f has three unequal roots.

(Clearly show the coordinates of the turning point and the point of inflection)

ACTIVITY 10

The function t is defined by $t(x) = 2x^3 + bx + c$ and has the following properties .

- $t(-3) = t(3) = t(0) = 0$
- $t'(-1{,}5) = t'(1{,}5) = 0$

Use this information to draw a neatly labelled sketch graph of t, without solving for b and c.

ACTIVITY 11

The following information about a cubic polynomial, $y = f(x)$ is given:

- $f(-1) = 0$
- $f(2) = 0$
- $f(1) = -4$
- $f(0) = -2$
- $f'(1) = f'(-1) = 0$
- if $x < -1$ then $f(x) > 0$
- if $x > 1$ then $f(x) > 0$

11.1 Use the information to draw a neat sketch of f

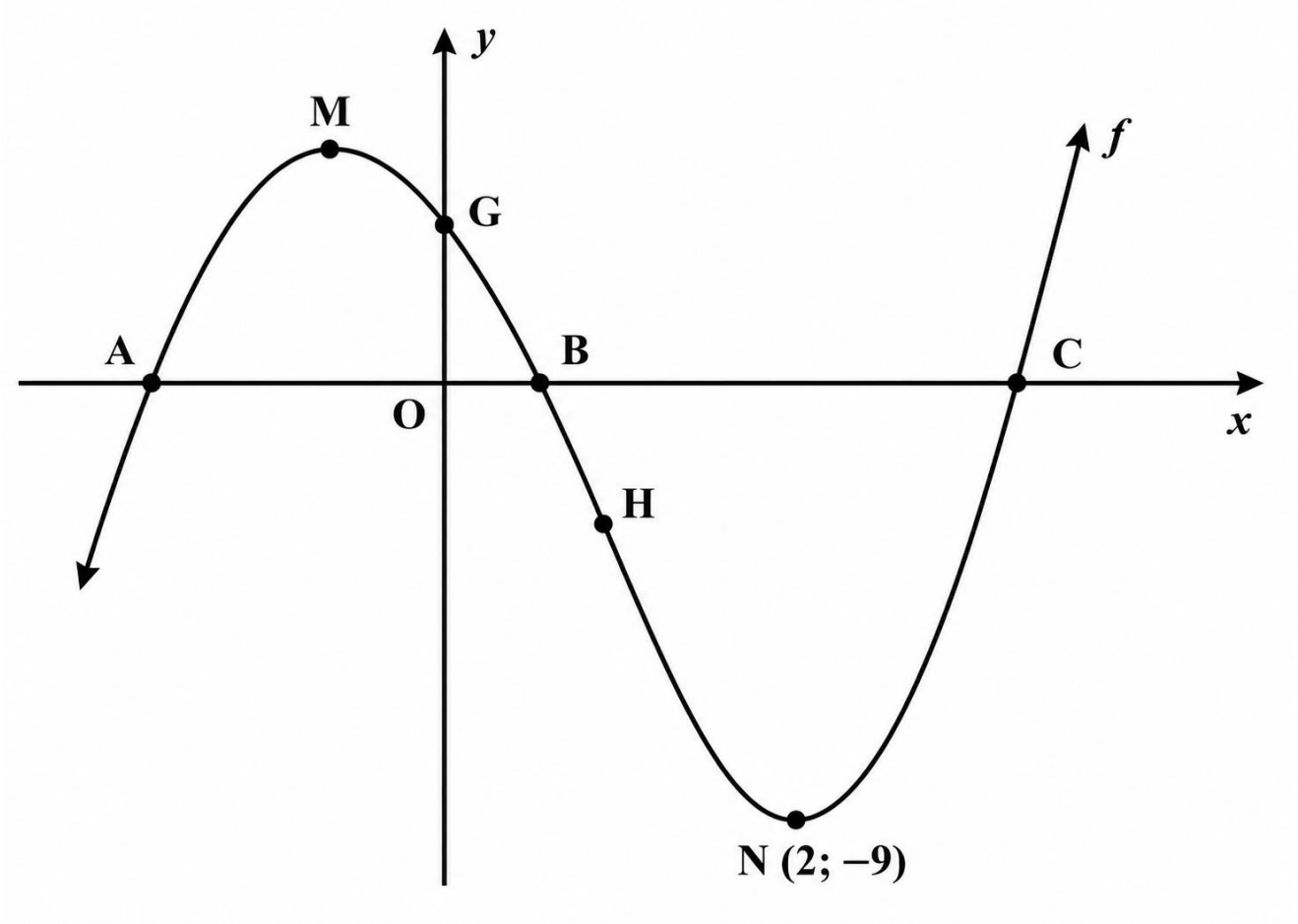
11.2 For which values of $x$is f increasing?

11.3 Use your graph to determine the x– coordinate of the point of inflection of f

11.4 For which values of x is f concave up?

ACTIVITY 12

The function $f(x) = 2x^3 + px^2 + qx + 3$ is drawn below. N$(2; -9)$ and M are turning points of f.
G is the y −intercept of f and A, B and C are the x- intercepts of f.



12.1 Show that $p = -5$ and $q = -4$

12.2 Determine the coordinates of G, the y-intercept of f.

12.3 If it is given that C(3;0), calculate the distance between A and B, the x-intercepts.

12.4 Calculate the x-coordinate of M, a turning point of f.

12.5 Determine the x-coordinate of H, the point of inflection of f.

12.6 For which value(s) of x will f be concave up?

12.7 For which value(s) of x is $f(x).f'(x) < 0$?

SECTION 2: ANALYTICAL GEOMETRY

2.1 Grade 11 revision

The **Distance** of the line **AB** between two points **A** and **B** will be given by;

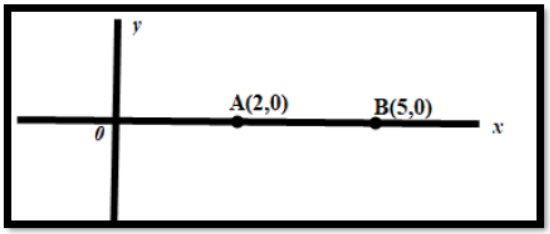
$$d_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

NOTE: If the line **AB** is **Vertical**, then we only consider the $y - values$ of the points, i.e.

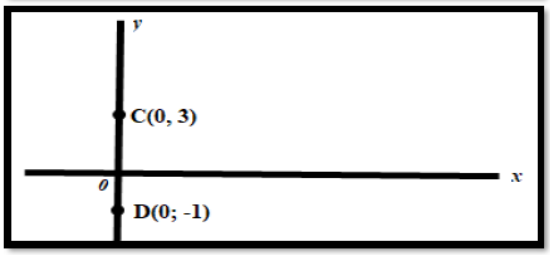
$$d_{CD} = y_{top} - y_{bottom}$$

If the line **AB** is **Horizontal**, then we only consider the $x - values$ of the points,

$$d_{AB} = x_{right} - x_{left}$$

- 

$AB = x_B - x_A$

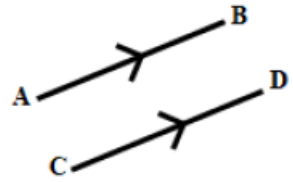
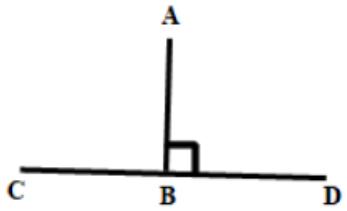
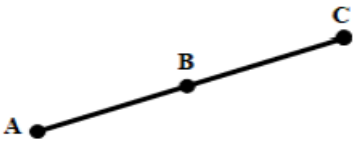
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$CD = y_C - y_D$

The standard form of the straight-line graph **($y = mx + c$),** as well as determining the equation of the line from the information given (sketch or co-ordinates of point/s). And to apply the alternate formula.

$y - y1 = m(x - x1)$ **and** $m = \frac{y_2 - y_1}{x_2 - x_1}$

- **Gradients**

A) PARALLEL LINES  $m_{AB} = m_{CD}$	**When given the equations for // lines** **Equation of the line AB** $y = 2x + 3$ $m_{AB} = 2$ **Equation of the line CD** $y - 2x = 5$ $y = 2x + 5$ $m_{CD} = 2$
B) PERPENDICULAR LINES  If $m_{AB} \times m_{CD} = -1$ AB ⊥ CD $A\hat{B}D = 90^{\circ}$	**When given the equations for ⊥ lines** **Equation of the line AB** $y = 2x + 1$ $m_{AB} = 2$ **Equation of the line CD** $2y = -x + 4$ $y = \frac{-1}{2}x + 2$ $m_{CD} = \frac{-1}{2}$ $m_{AB} \times m_{CD} = -1$
C) COLLINEAR POINTS  $m_{AB} = m_{BC} = m_{AC}$	Calculate the value of q if $K(-6\,;\,9)$, $L(-3\,;\,q)$ and M(–2 ; –1) are collinear.

When the gradients of lines are equal, then the lines are parallel.

($m_{AB} = m_{CD}$; ∴ AB ∥ CD)

If the product of the gradients are equal to -1 , then the two lines are perpendicular to each other.

$m_{AB} \times m_{CD} = -1$= ∴ **AB ⊥ CD)**

- Angle of Inclination

Another alternative to determine the **gradient** of a line, is to use the **Angle of Inclination**.:

$$\tan\theta = m_{line}$$

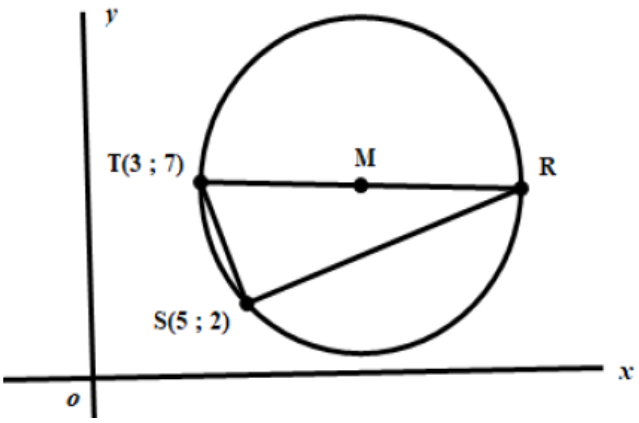
- The **Midpoint M** of the line **AB** will be given by;

$$M(x_M = \frac{x_A + x_B}{2}; y_M \frac{y_A + y_B}{2})$$

Where $(x_M; y_M)$ is the x *and* y values of the midpoint.

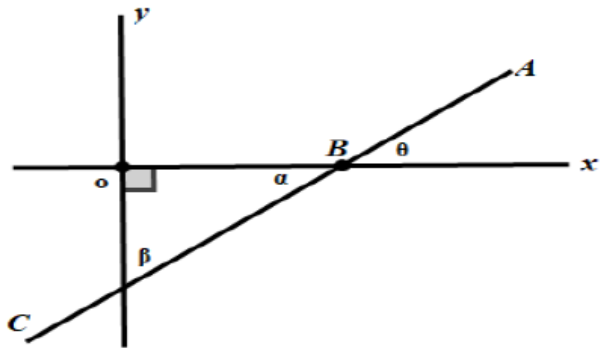
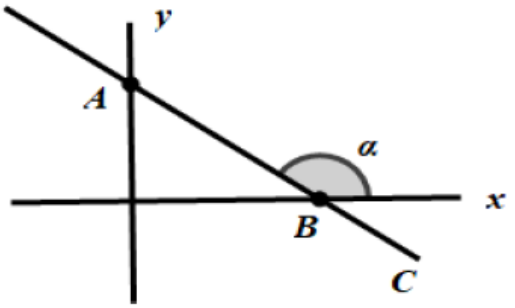
$\mathbf{M}\left(\frac{x_1 + x_2}{2} ; \frac{y_1 + y_2}{2}\right)$

- **Determine the midpoint of TS.**
- **If $\mathbf{M}\left(9 ; 6\frac{1}{2}\right)$ the midpoint of TR, Calculate the coordinates of R.**

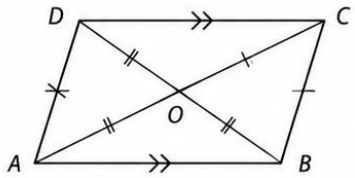
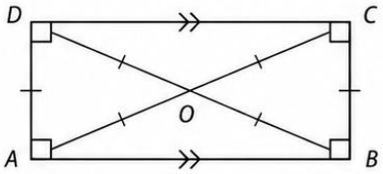
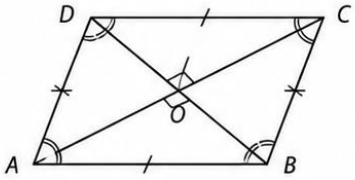
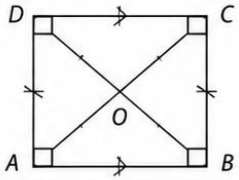
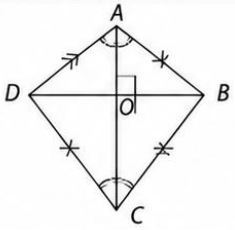
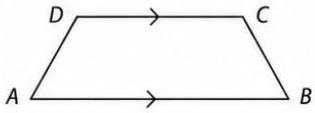


5. ANGLE OF INCLINATION

- $\boldsymbol{tan\theta = m}$ where **m** is the gradient.
- It is always between **0° and 180°**.
- An angle measured from the horizontal line or from x- axis. If it is not from the horizontal line, **construct line** from horizontal line to get required angle of inclination.

If $m > 0$ $\angle$ of inclination $0° < \theta < 90°$	If $m < 0$ $\angle$ of inclination $90° < \theta < 180°$
 ➢ $tan\theta = m_{AB}$ $\theta = tan^{-1}(m_{AB})$ **acute** $\angle$ ➢ $\boldsymbol{\theta = \alpha}$ **(vertical opposite** $\boldsymbol{\angle' s}$**)**	 ➢ $tan\alpha = m_{AB}$ $\alpha = tan^{-1}(+ m_{AB})$ **obtuse** $\angle$ $\boldsymbol{\alpha = 180° - ref\ \angle}$ ✓ If you get angle of inclination as a negative angle just add 180°

You needs to learn the following properties of the following quadrilaterals

PROPERTIES OF SPECIAL QUADRILATERALS	
PARALLELOGRAM • Both pairs of opposite sides are parallel • Both pairs of opposite sides are equal • Both pairs of opposite angles are equal • Diagonals bisect each other	
RECTANGLE All properties of parallelogram Plus: • Both diagonals are equal in length • All interior angles are equal to 90°	
RHOMBUS All properties of parallelogram Plus: • All sides are equal • Diagonals bisect each other perpendicularly • Diagonals bisect interior angles	
SQUARE All properties of a rhombus Plus: • All interior angles are 90° • Diagonals are equal in length	
KITE • Two pairs of adjacent sides are equal • Diagonal between equal sides bisects other diagonal • One pair of opposite angles are equal (unequal sides) • Diagonal between equal sides bisects interior angles (is axis of symmetry) • Diagonals intersect perpendicularly	
TRAPEZIUM • One pair of opposite sides are parallel	

WORKED EXAMPLE 1

1. If the points $A(6;5)$, $B(3;2)$ and $C(2k;k+4)$ are given such that $AB \perp BC$, then calculate the value of k.

2. $P(2\ ;\ 8)$, $Q(4\ ;\ y)$ and $R(x\ ;\ 0)$ are given points on line segment PR. Determine x and y if Q is the midpoint of PR.

3. The points $A(1\ ;\ 2)$, $B(p-1\ ;\ 3)$ and $C(-3\ ;\ -6)$ are collinear. Determine p.

Solution

1.

$m_{AB} \times m_{BC} = -1$

$\therefore \frac{2-5}{3-6} \times \frac{(k+4)-2}{2k-3} = -1$

$\therefore \frac{-3}{-3} \times \frac{k+2}{2k-3} = -1$

$\therefore 1 \times \frac{k+2}{2k-3} = -1$

$\therefore \frac{k+2}{2k-3} = -1$

$\therefore k + 2 = -2k + 3$

$\therefore 3k = 1$

$\therefore k = \frac{1}{3}$

2.

$4 = \frac{2+x}{2}$ and $y = \frac{0+8}{2}$

$\therefore x = 6$ and $y = 4$

3.

$m_{AB} = m_{BC}$

$\therefore \frac{3-2}{p-1-1} = \frac{-6-3}{-3-(p-1)}$

$\therefore \frac{1}{p-2} = \frac{-9}{-3-p+1}$

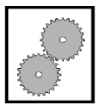
$\therefore \frac{1}{p-2} = \frac{-9}{-p-2}$

$\therefore -p - 2 = -9(p-2)$

$\therefore -p - 2 = -9p + 18$

$\therefore 8p = 20$

$\therefore p = \frac{20}{8} = \frac{5}{2}$



WORKED EXAMPLE 2

The equation of a line is defined by $y = (2k - 3)x + 12$.

Find the value of *k* if:

2.1 the line is parallel to $y = 4x + 7$.

2.2 the line is parallel to the *x*-axis.

2.3 the line passes through the point $(-3\ ;\ 4)$.

Solution

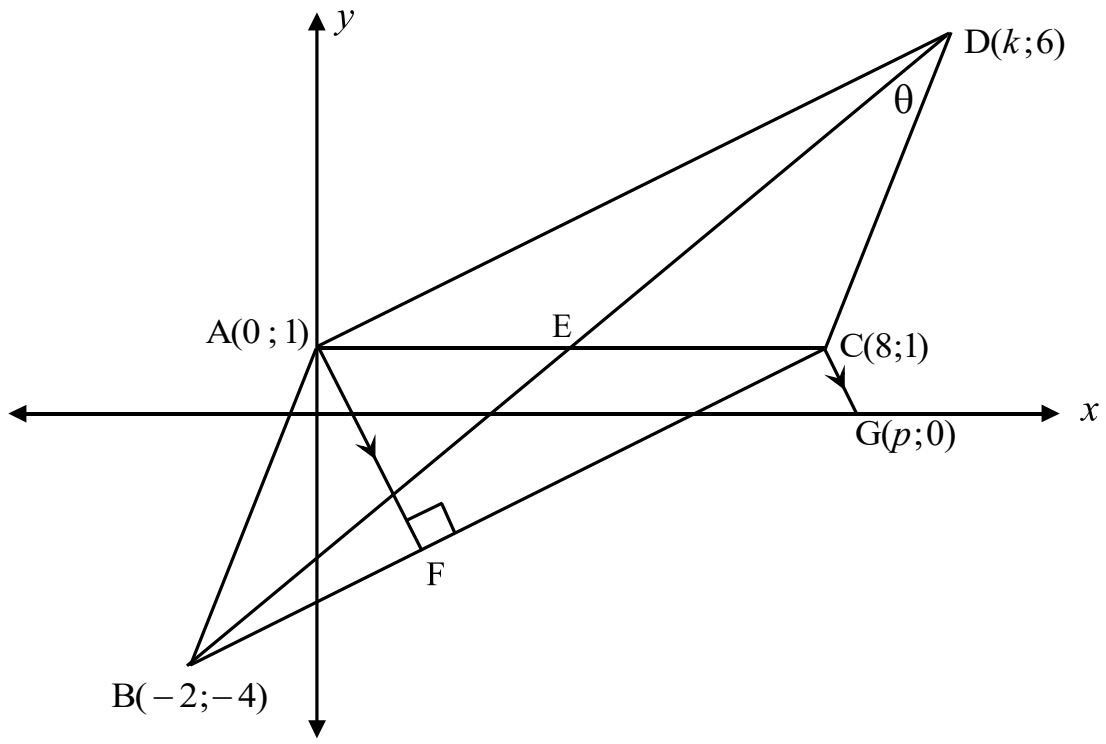
2.1 $2k - 3 = 4$
$\therefore 2k = 7$
$\therefore k = \frac{7}{2}$

2.2 $2k - 3 = 0$
$\therefore 2k = 3$
$\therefore k = \frac{3}{2}$

2.3 $4 = (2k - 3)(-3) + 12$
$\therefore 4 = -6k + 9 + 12$
$\therefore 6k = 17$
$\therefore k = \frac{17}{6}$

WORKED EXAMPLE 3

In the figure below, ABCD is a parallelogram with vertices $A(0\ ;\ 1)$, $B(-2\ ;\ -4)$, $C(8\ ;\ 1)$ and $D(k\ ;\ 6)$. AF is perpendicular to BC and parallel to CG. E is the point of intersection of the diagonals of ABCD.



Determine:

3.1 the length of BC in simplest surd form.

3.2 the gradient of BC.

3.3 the equation of AF.

3.4 the coordinates of E.

3.5 the value of *k*.

3.6 the value of *p*.

3.7 the size of θ rounded off to two decimal places.

Solution

3.1 $\text{BC}^2 = (8-(-2))^2 + (1-(-4))^2$

$\therefore \text{BC}^2 = 125$

$\therefore \text{BC} = \sqrt{125} = 5\sqrt{5}$

3.2 $m_{\text{BC}} = \dfrac{1-(-4)}{8-(-2)} = \dfrac{5}{10} = \dfrac{1}{2}$

3.3 $m_{\text{AF}} = -2$

$y-$ int of AF is 1

$\therefore y = -2x + 1$

3.4 $x_E = \dfrac{0+8}{2} = 4 y_E = \dfrac{1+1}{2} = 1$

$E(4\ ;\ 1)$

3.5 $4 = \dfrac{-2+k}{2}$

$\therefore 8 = -2 + k$

$\therefore k = 10$

3.6 $m_{\text{CG}} = m_{\text{AF}} = -2$

$\therefore \dfrac{0-1}{p-8} = -2$

$\therefore -1 = -2(p-8)$

$\therefore -1 = -2p + 16$

$\therefore 2p = 17$

$\therefore p = \dfrac{17}{2}$

3.7 $tan\,\alpha = m_{\text{DC}} = \dfrac{6-1}{10-8} = \dfrac{5}{2}$

$tan\,\alpha = \dfrac{5}{2}$

$\therefore \alpha = 68{,}19859051°$

$tan\,\beta = m_{\text{DB}} = \dfrac{6-(-4)}{10-(-2)} = \dfrac{10}{12} = \dfrac{5}{6}$

$tan\,\beta = \dfrac{5}{6}$

$\therefore \beta = 39{,}80557109°$

$\alpha = \theta + \beta$

$\therefore \alpha - \beta = \theta$

$\therefore \theta = 68{,}19859051° - 39{,}80557109°$

$\therefore \theta = 28{,}39°$

THE CIRCLES ANALYSIS

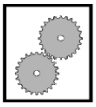
The equation of a circle with **centre** $(a; b)$ **and radius** r **is** $(x-a)^2+(y-b)^2=r^2$

The above equation is in **centre form.** Another form of a circle's equation with any centre is

$ax^2+by^2+cx+dy+e=0$ which is a **general form.**

By multiplying out the equation in centre-form results in the equation in general form.

Similarly by manipulating the equation in general form by **completing the square** for both x and y results in the equation back to centre form.



WORKED EXAMPLE 4

1. Determine the co-ordinates of the centre and the length of the radius for each of the following circles.

a) $(x-5)^2+(y+3)^2=9$

Centre: $(5;-3)$ and $r^2=9$

$r=3$

2. Given $x^2+y^2+2x-6y-6=0$, find the centre and radius of this circle.

$x^2+2x+y^2-6y=6$

Complete the square for the expression in x and y separately :

$$x^2+2x+\left(2\times\frac{1}{2}\right)^2+y^2-6y+\left(-6\times\frac{1}{2}\right)^2=6+\left(2\times\frac{1}{2}\right)^2+\left(-6\times\frac{1}{2}\right)^2$$

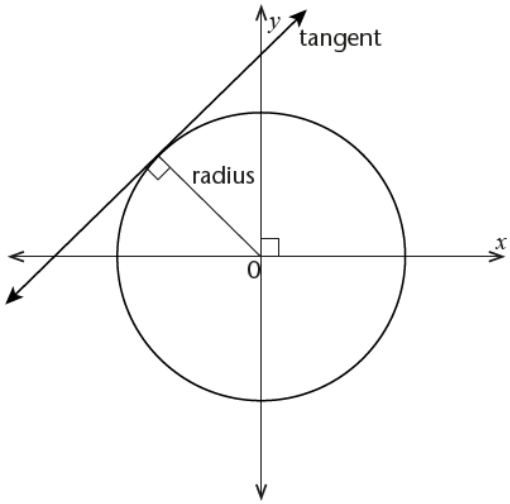
$x^2+2x+1+y^2-6y+9=16$

$(x+1)^2+(y-3)^2=16$

Centre: $(-1;3)$ and $r^2=16, \therefore r=4$

THE EQUATION OF A TANGENT

A tangent is a line that touches the circle at only one point. Even if it's extended, it will never touch the circle again.



- Tangent is always perpendicular to the radius at the point of contact. This means the angle between the radius and the tangent is 90^o

Therefore, $m_{rad} \times m_{tan} = -1 \ (rad \perp tan)$.

WORKED EXAMPLE 5

Determine the equation of the tangent to the circle defined by $x^2 + y^2 = 2$ at a point $(-1\ ;\ 1)$.

1. $m_{rad} = \frac{y_2 - y_1}{x_2 - x_1}$

$$m_{rad} = \frac{0 - 1}{0 - (-1)}$$

$$m_{rad} = -1$$

$$m_{tan} = \frac{-1}{m_{rad}}$$

$$m_{tan} = \frac{-1}{-1} \qquad \therefore \ m_{tan} = 1$$

$$y - y_1 = m\,(x - x_1)$$

$$y - 1 = 1\,(x - (-1))$$

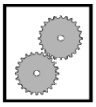
$$y - 1 = x + 1$$

$$y = x + 2$$

POINT OF INTERSECTION

The point of intersection of a circle and a straight line is a point at which the circle and the line meet (where they are equal). We solve equations simultaneously to find the coordinates of the point of intersection.

- A line segment that cuts through the circle (two points of intersection) is called a secant.
- A line segment that touches (one point of intersection) the circle is called a tangent



WORKED EXAMPLE 6

Determine the point(s) of intersection of the line $x + y = 2$ and the circle $x^2 + y^2 = 2$

Solution:

$x + y = 2$ ……..…eq 1

$x^2 + y^2 = 2$…….…. eq 2

From eq 1 $x = 2 - y$ ….…eq 3

Subs eq 3 into eq 2

$(2 - y)^2 + y^2 = 2$

$4 - 4y + y^2 + y^2 - 2 = 0$

$2y^2 - 4y + 2 = 0$

$y^2 - 2y + 1 = 0$

$(y - 1)(y - 1) = 0$

$y = 1$

$x = 1 \qquad (1\ ;\ 1)$

2.1 The Intersection of circles analysis

To determine whether there is an intersection between two circles or not. If they intersect internally, at one point or externally, we need to determine the distance (d) between the centers of the two circles in comparison to the sum $(r_1 + r_2)$ of the radius of these two circles.

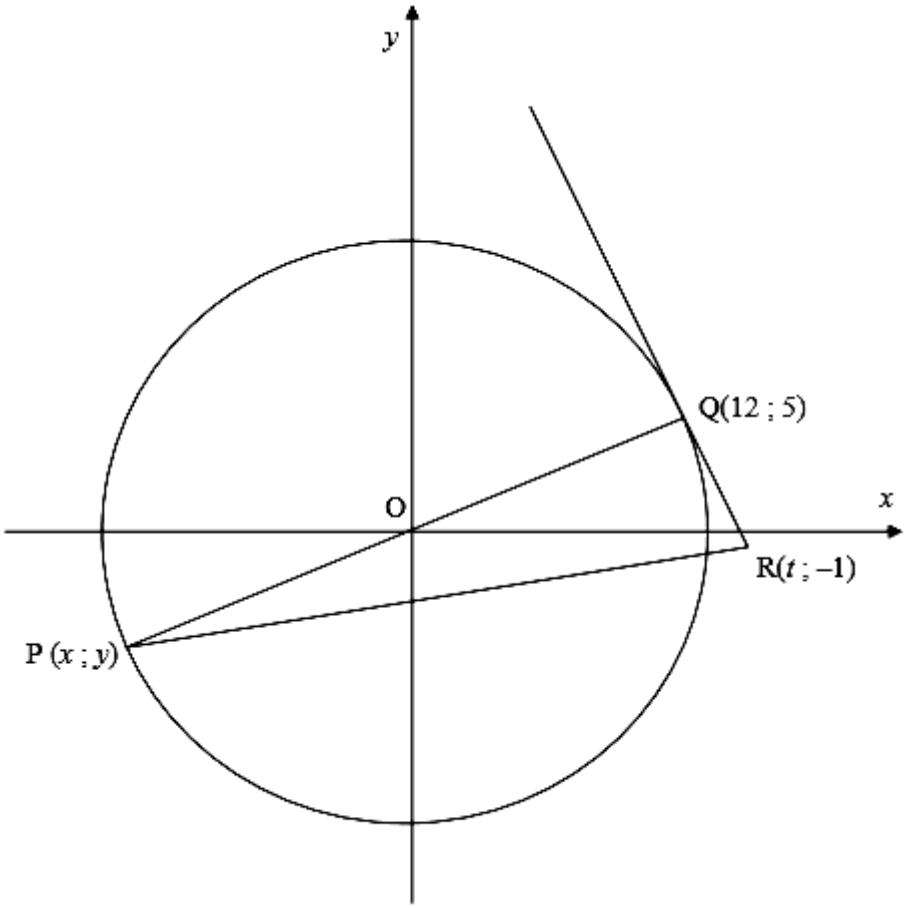
- If $d < r_1 + r_2$, then the two circles intersect internally.
- If $d = r_1 + r_2$, then the two circles intersect at only one point or intersect externally.
- If $d > r_1 + r_2$, then there is no intersection between the two circles.
-

Sometimes you will be asked to determine if a point, p lies inside, outside or on the circle . We need to find the distance (d) between center and that point.

- If $d < r$, then the point is inside the circle.
- If $d = r$, then the point is on the circumference of the circle.
- If $d > r$, then the point lies outside the circle.

WORKED EXAMPLE 7

O is the centre of the circle in the figure below. $P(x;y)$ and $Q(12;5)$ are two points on the circle. POQ is a straight line. The point $R(t;-1)$ lies on the tangent to the circle at Q.



7.1 Determine the equation of the circle.

7.2 Determine the equation of the straight line through P and Q.

7.3 Determine x and y, the coordinates of P.

7.4 Show that the gradient of QR is $-\frac{12}{5}$.

7.5 Determine the equation of the tangent QR in the form $y =\ldots$

7.6 Calculate the value of t.

7.7 Determine an equation of the circle with centre $Q(12;5)$ and passing through the origin

Solution

7.1 $d_{OQ} = \sqrt{(12-0)^2 + (5-0)^2}$

$r = \sqrt{169} = 13$

$\therefore (x-0)^2 + (y-0)^2 = (13)^2$

$x^2 + y^2 = 169$

7.2 $m_{PQ} = \frac{5-0}{12-0} = \frac{5}{12}$

$\therefore y = \frac{5}{12}x$

7.3 P(-12;-5)

7.4 $m_{rad} \times m_{tan} = -1 \qquad (rad \perp tan)$

$\frac{5}{12} \times m_{tan} = -1$

$m_{tan} = -\frac{12}{5}$

7.5 $y - y_Q = mQ_{tan}$

$y - 5 = -\frac{12}{5}(x-12)$

$y = -\frac{12}{5}x + \frac{169}{5}$

7.6 $y = -\frac{12}{5}x + \frac{169}{5}$

substitute R, the point on the line

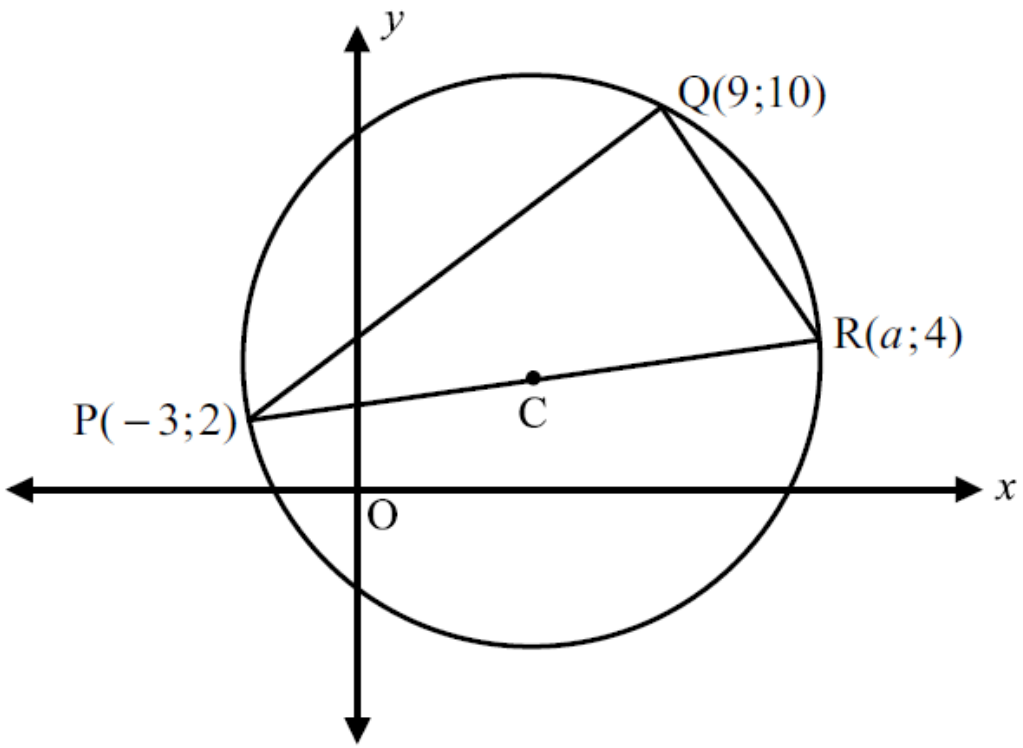
$-1 = -\frac{12}{5}t + \frac{169}{5}$

$\therefore t = \frac{29}{2} = 14.5$

7.7 $(x-12)^2 + (y-5)^2 = 169$

WORKED EXAMPLE 8

The points $P(-3\ ;\ 2)$, $Q(9\ ;\ 10)$ and $R(a\ ;\ 4)$ lie on the circumference of the circle, as shown in the diagram below. PR is a diameter of the circle with centre C.



8.1.1 Show that $a = 13$.

8.1.2 Determine the equation of the circle in the form $(x-a)^2 + (y-b)^2 = r^2$.

8.2 The equations of two circles with centres A and B are given below.

Circle A: $(x-2)^2 + (y-3)^2 = 9$

Circle B: $(x-1)^2 + (y+1)^2 = 16$

Without solving for *x* and *y*, determine if the circles intersect each other in two points, touch each other, or do not intersect each other. Show all your calculations.

Solution

8.1.1 $\hat{Q} = 90°$ ∠ in semi-circle

$m_{PQ} \times m_{QR} = -1$

$\therefore \frac{10-4}{9-a} \times \frac{10-2}{9-(-3)}$

$\therefore \frac{6}{9-a} \times \frac{8}{12} = -1$

$\therefore \frac{6}{9-a} \times \frac{2}{3} = -1$

$\therefore \frac{12}{27-3a} = -1$

$\therefore 12 = -27 + 3a$

$\therefore 39 = 3a$

$\therefore a = 13$

8.1.2 Centre: $C\left(\frac{-3+13}{2} ; \frac{2+4}{2}\right) = (5 ; 3)$

Radius: $r^2 = (5-(-3))^2 + (3-2)^2 = 65$

$\therefore r = \sqrt{65}$

Equation of circle:

$(x-5)^2 + (y-3)^2 = 65$

8.2 $r_A = 3$ and $A(2 ; 3)$

$r_B = 4$ and $B(1 ; -1)$

Distance between A and B:

$AB^2 = (1-2)^2 + (-1-3)^2 = 17$

$AB = \sqrt{17} = 4{,}1231\ldots$

$r_A + r_B = 7 > \sqrt{17}$

The circles intersect each other.

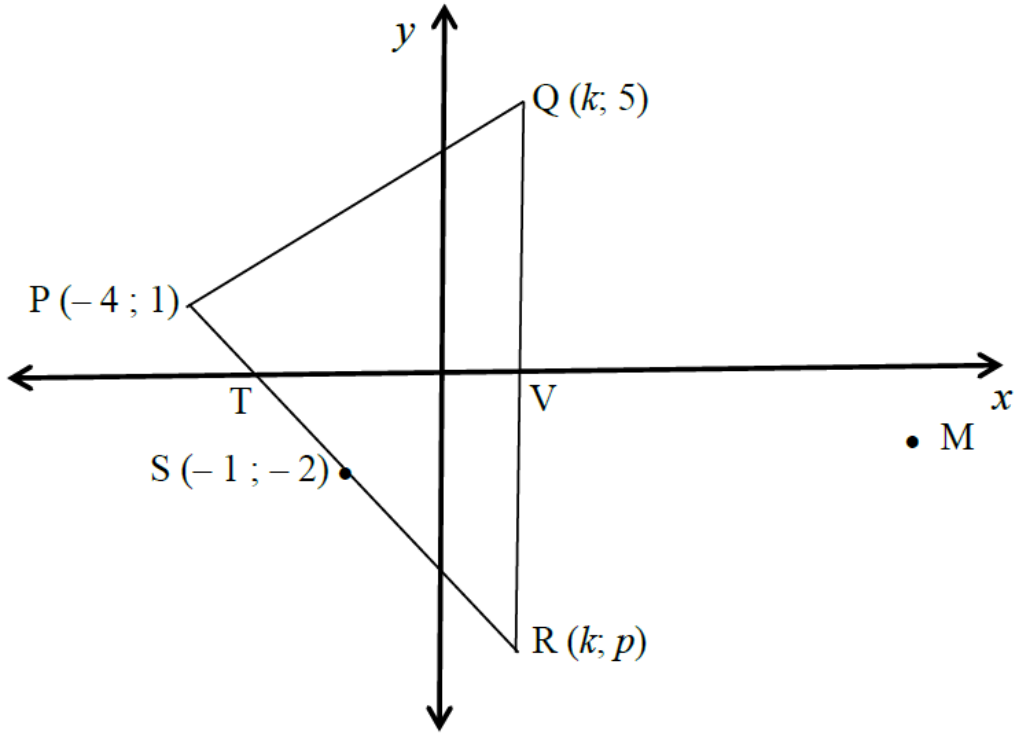
ACTIVITIES

SECTION 2: ANALYTICAL GEOMETRY

ACTIVITY 1

In the diagram below, P (– 4; 1), Q (k; 5) and R(k; p) are vertices of ΔPQR.

S (– 1; – 2) is a midpoint of PR. T is the x-intercept of PR and V is the x-intercept of QR. QR is perpendicular to the x-axis



1.1 Show that $k = 2$ and $p = -5$.

1.2 Calculate the length of PR.

1.3 Determine the gradient of PR.

1.4 Determine the equation of PR, in the form $y = mx + c$.

1.5 Determine whether or not, the point (– 11; 14) lies on the line determined in Question 1.4.

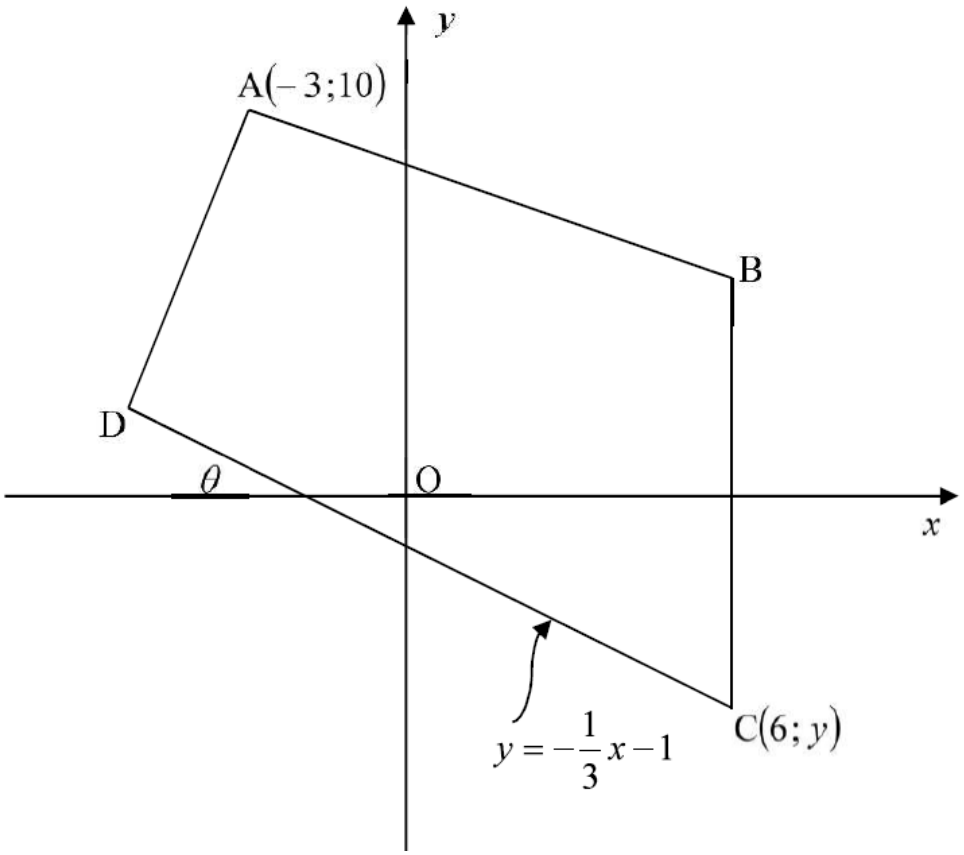
1.6 Determine the equation of the line that passes through point R and is perpendicular to line PR.

1.7 Determine the coordinates of M if PQMR is a parallelogram.

1.8 Calculate the size of $P\hat{R}Q$

ACTIVITY 2

In the diagram below, point A $(-3; 10)$ and B in the first quadrant, are points on quadrilateral ABCD. Equation of line DC is given as $y = -\frac{1}{3}x - 1$ and DC form an angle θ with the x-axis as indicated.



2.1 Write down the gradient of DC.

2.2 Calculate the size of θ.

2.3 Determine the equation of the line AD if $AD \perp DC$.

2.4 Determine the y −coordinate of C.

2.5 Determine:

2.5.1 The coordinates of D and hence calculate the length of AD.

2.5.2 The area of ΔACD

2.6 Prove whether quadrilateral ABCD is a parallelogram if it is given that coordinates Of B $(6; 6)$. Show all necessary calculations.

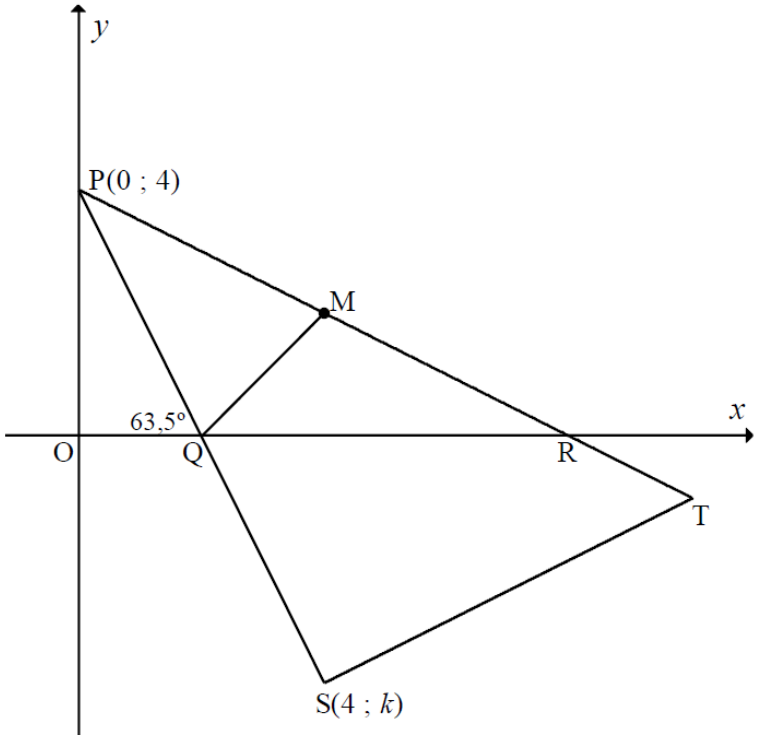
ACTIVITY 3

In the diagram below, triangle PST, with vertices P $(0; 4)$, S $(4; k)$ and T, is shown.

Q and R are the x-intercepts of PS and PT respectively. $OQ : OR = 1 : 4$.

M is the midpoint of PR.

$0\widehat{Q}P = 63{,}5°$



3.1 Determine the gradient of PS, correct to **the nearest integer**.

3.2 Write down the equation of PS.

3.3 Show that $k = 4$.

3.4 Determine the length of QR.

3.5 Calculate the coordinates of M.

3.6 Prove that QM cannot be the perpendicular bisector of PR.

3.7 Calculate the area of quadrilateral QSTM, if the coordinates of T are $(10, -1)$.

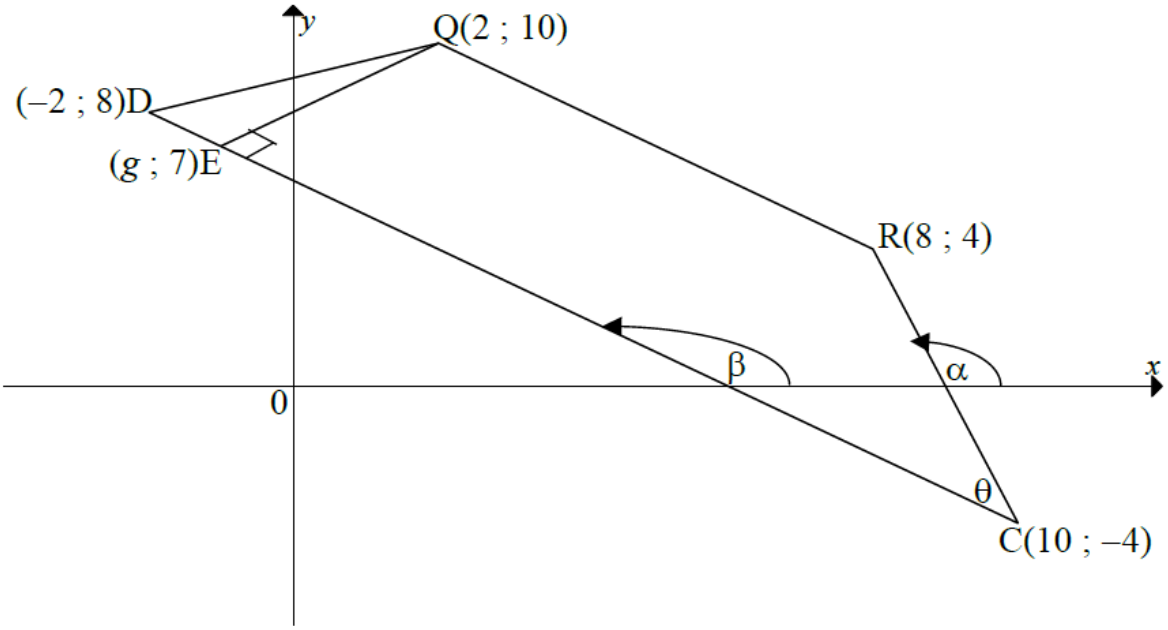
ACTIVITY 4

QRCD is a trapezium with vertices M(2 ; 10), R(8 ; 4), C(10 ; – 4) and D(– 2 ; 8).

Equation of line CD is given by $y = -x + 6$.

The angles of inclination of CD and RC are β and α respectively.

E$(g\ ;\ 7)$ is a point on CD such that QE⊥CD. $R\hat{C}D = \theta$



4.1 Show that $g = -1$.

4.2 Determine the length of CD.

4.3 Determine the equation of RC, in the form $y = ...$

4.4 Calculate the size of angle θ.

4.5 Point M, the midpoint of CD, such that QRCM is a parallelogram.

Determine the coordinates of M.

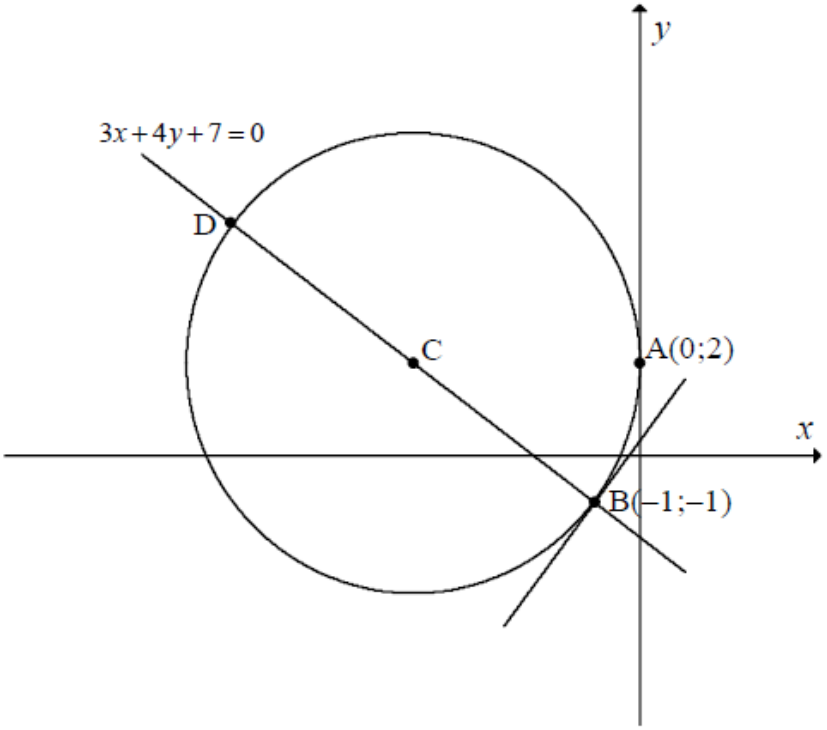
4.6 Area of trapezium: $A = \frac{1}{2}(a + b)h$.

Calculate the area of trapezium QRCD if it is given that QR = $6\sqrt{2}$

4.7 If $QD = \sqrt{20}$ and $QM = \sqrt{68}$, calculate the perimeter of ΔDMQ.

ACTIVITY 5

5. 1 In the diagram below, the circle with centre C touches the *y*-axis at A(0; 2).
A straight line with equation $3x + 4y + 7 = 0$ cuts the circle at B$(-1; -1)$ and D.



5.1.1 Determine the equation of the tangent to the circle at B.

5.1.2 Determine the equation of the circle in the form $(x - a)^2 + (y - b)^2 = r^2$.

5.1.3 Determine the coordinates of the image of B, after reflection of the circle in the line $y = 2$.

5.2 A circle with equation $x^2 - 4x + y^2 + 6y - 54 = 0,$ is drawn in a Cartesian plane.

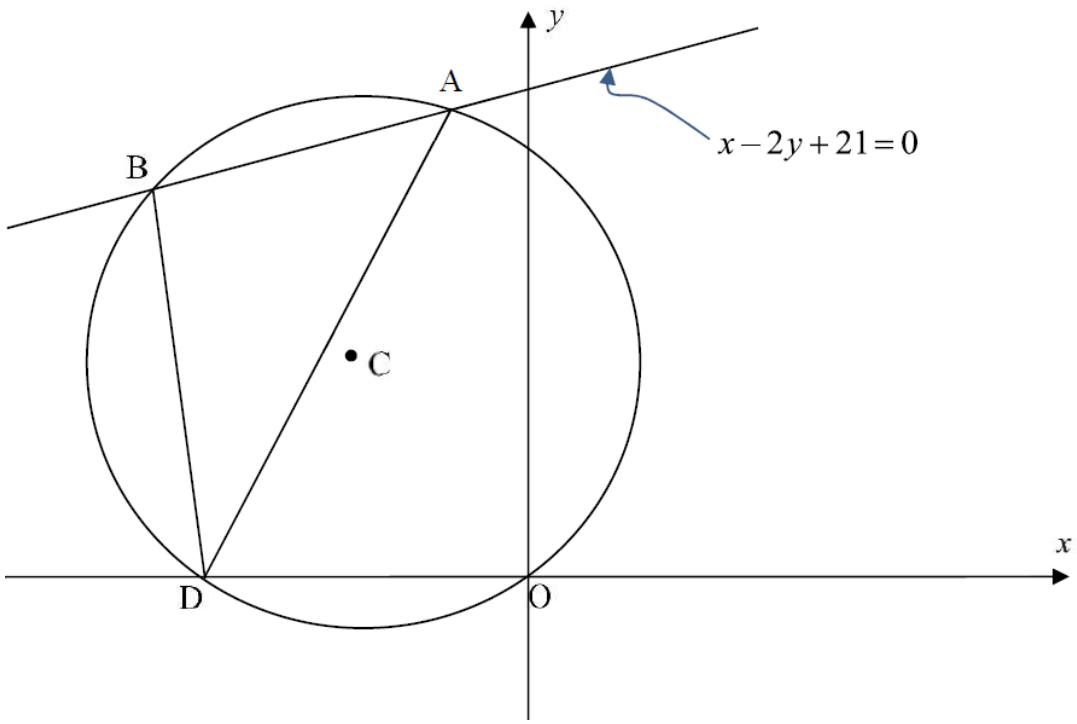
5.2.1 Determine the coordinates of the centre of the circle and the length of its radius.

5.2.2 Another circle with equation $x^2 + y^2 = r^2$ is drawn in the same Cartesian plane and touches the circle with equation $x^2 - 4x + y^2 + 6y - 54 = 0$ internally.

Calculate the value of *r*. Give your answer correct to 2 decimal digits.

ACTIVITY 6

In the diagram below, A, B, D and the origin O lie on the circle with centre C. The equation of the circle is $x^2 + y^2 + 6x - 8y = 0$. The equation of AB is $x - 2y + 21 = 0$.



6.1 Determine the coordinates of the centre C, and the radius of the circle.

6.2 Determine the coordinates of A and B.

6.3 Determine:

6.3.1 The midpoint of AD.

6.3.2 Whether CB passes through the midpoint of line AD.

6.4 A second circle with equation $(x - 3)^2 + (y + 4)^2 = 25$ is drawn on the same set of axes. A student claims that the circles touch each other in one point. Is the student correct? Show all calculations.

ACTIVITY 7

7.1 Determine the centre and radius of the circle with the equation $x^2 + y^2 + 8x + 4y - 38 = 0$.

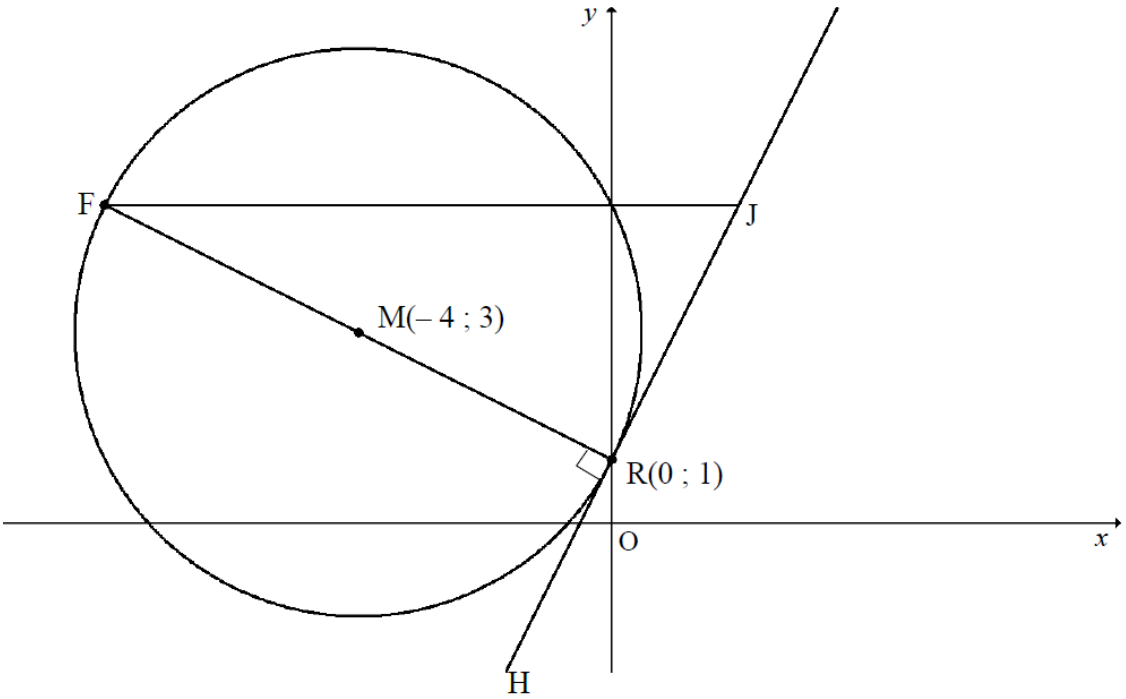
7.2 A second circle has the equation $(x - 4)^2 + (y - 6)^2 = 26$. Calculate the distance between the centres of the two circles.

7.3 Hence, show that the circles described in QUESTION 7.1 and QUESTION 7.2 intersect each other.

7.4 Show that the two circles intersect along the line $y = -x + 4$.

ACTIVITY 8

Drawn below is a circle with centre M(– 4 ; 3).
R(0 ; 1) and F(*x ; y*) are two points on the circle such that FR is a diameter of the circle.
JRH is a tangent to the circle at R. The ratio of FJ : JR = 2,237 : 1.



8.1 Determine the equation of the circle with centre M, in the form $(x-a)^2+(y-b)^2=r^2$

8.2 Determine the equation of tangent JRH to the circle at R in the form $y = \ldots$

8.3 Does point P(5 ; 2) lie inside, outside or on the circle? Justify your answer.

8.4 FJ is parallel to the x-axis

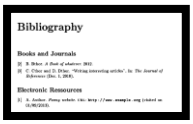
8.4.1 Determine the coordinates of J.

8.4.2 Calculate the length of FJ.

8.4.3 Hence, write down the length of JR.

8.4.4 If J is reflected about the line $y = x$, and translated 2 units upwards, determine the coordinates of J' after transformation.

8.5 If $g(x) = k$, for which values of k will g cut the circle exactly twice?



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