JENN

Training and Consultancy

The path to enlightened education

SUBJECT: TECHNICAL MATHEMATICS

GRADE 12

WINTER CLASSES

TEACHER AND LEARNER CONTENT MANUAL

Topic(s)

1. ALGEBRA
2. ANALYTICAL GEOMETRY
3. EUCLIDEAN GEOMETRY
4. CALCULUS

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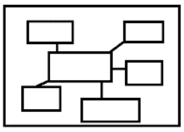
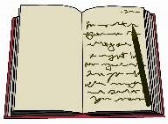
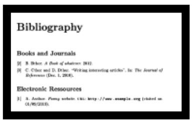
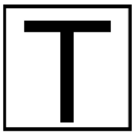
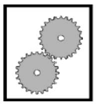
ICON DESCRIPTION

 MIND MAP	 EXAMINATION GUIDELINE	 CONTENTS	 ACTIVITIES
 BIBLIOGRAPHY	 TERMINOLOGY	 WORKED EXAMPLES	 STEPS

CONTENTS PAGE

SECTION 1: ALGEBRA

1. EQUATIONS

1.1 Quadratic Equations are equations of the second degree (i.e. the highest exponent of the variable is 2).The standard form of quadratic equation is $ax^2 + bx + c = 0$ where $a \neq 0$.

QUADRATIC FORMULA	DIFFERENCE OF TWO SQUARES	COMPLETE THE SQUARE	ONE ROOT	TWO ROOTS	FRACTIONS AND RESTRICTIONS
Substitute into the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where a = coefficient of x^2, b = coefficient of x, c = constant term		1. Write in standard form 2. Move C across 3. Divide both sides by A 4. Add (½ x b)² to both sides 5. Factorise and solve	1. Substitute the known root 2. Solve for the variable 3. Substitute the value of the variable and solve for the root	1. Substitute the roots into the equation 2. Use "FOIL" for the quadratic equation	1. Find the LCD and list restrictions 2. Solve for x 3. Check your answers against your restrictions REMEMBER: $\frac{0}{x} = 0$ BUT $\frac{x}{0} = undefined$
<u>Eq. $-3x^2 = -12 + 7x$</u> $-3x^2 - 7x + 12 = 0$ $a = -3;\ b = -7;\ c = 12$ $x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(-3)(12)}}{2(-3)}$ $x = \frac{7 \pm \sqrt{49 + 144}}{-6}$ $x = \frac{7 \pm \sqrt{193}}{-6}$ $x = \frac{7 + \sqrt{193}}{-6}$ or $x = \frac{7 - \sqrt{193}}{-6}$ Answer in surd form or can be calculated/rounded off to 2 decimals x = −3,48 OR x = 1,15	Either method may be used <u>Eq. $x^2 = 25$</u> $x^2 - 25 = 0$ $(x - 5)(x + 5) = 0$ x = 5 or x = −5 Therefore x = ±5 $\sqrt{x^2} = \pm\sqrt{25}$ x = ±5	<u>Eq. $x^2 + 2x = 1$</u> $(\frac{1}{2} . 2)^2$ (1) $x^2 + 2x + 1 = 1 + 1$ $\sqrt{(x+1)^2} = \pm\sqrt{2}$ $x + 1 = \pm\sqrt{2}$ $x + 1 = -\sqrt{2}$ or $x + 1 = \sqrt{2}$ $x = -1 - \sqrt{2}$ $x = -1 + \sqrt{2}$	<u>Eq. $x^2 + mx - 15 = 0$, where 5 is a root.</u> $(5)^2 + (5)m - 15 = 0$ $25 + 5m - 15 = 0$ $5m = -10$ $m = -2$ $x^2 - 2x - 15 = 0$ $(x - 5)(x + 3) = 0$ x = 5 or x = −3 (given)	<u>Eq. −9 and 7 are the roots of a quadratic equation</u> x = −9 or x = 7 x + 9 = 0 x − 7 = 0 $(x + 9)(x - 7) = 0$ $x^2 - 7x + 9x - 63 = 0$ $x^2 + 2x - 63 = 0$	<u>Eq.</u> $\frac{x}{x-2} = \frac{1}{x-3} - \frac{2}{2-x}$ $\frac{x}{x-2} = \frac{1}{x-3} + \frac{2}{x-2}$ LCD: $(x-2)(x-3)$ Restrictions: $x - 2 \neq 0;\ x - 3 \neq 0$ $\frac{x\cancel{(x-2)}(x-3)}{\cancel{(x-2)}} = \frac{1(x-2)\cancel{(x-3)}}{\cancel{(x-3)}} + \frac{2\cancel{(x-2)}(x-3)}{\cancel{(x-2)}}$ $x(x-3) = 1(x-2) + 2(x-3)$ $x^2 - 3x = x - 2 + 2x - 6$ $x^2 - 3x - x - 2x + 2 + 6 = 0$ $x^2 - 6x + 8 = 0$ $(x - 4)(x - 2) = 0$ x − 4 = 0 or x − 2 = 0 x = 4 x = 2 Check restrictions: x ≠ 2, x ≠ 3 Thus, x = 4 is the only solution.

WORKED EXAMPLE 1

Solve for x, rounded off to TWO decimal places where necessary:

1. $x^2 = 5x - 4$
2. $x(3 - x) = -3$
3. $x(x - 4) = 5$
4. $4x^2 - 20x + 1 = 0$

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Solutions

1 $x^2 = 5x - 4$

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$x = 4$ or $x = 1$,

2 $x(3-x) = -3$

$3x - x^2 = -3$

$x^2 - 3x - 3 = 0$

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-3)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{21}}{2}$$

$x = 3{,}79$ or $x = -0{,}79$

3 $x(x-4) = 5$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$x = 5$ or $x = -1$

4 $4x^2 - 20x + 1 = 0$

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(4)(1)}}{2(4)}$$

$$x = \frac{20 \pm \sqrt{384}}{8}$$

$x = 4{,}95$ or $x = 0{,}05$

1.2 Exponential equations

Solving equations where x is part of the exponent:

- Write the powers as products of prime factors.
- Aim to get ONE power with the same base on each side of the equation.
- Equate the exponents.
- Solve for x.

WORKED EXAMPLE 2

Solve for x:

2.1 $2^x = 8$

2.2 $5^{2x+1} - 125^{2x-3} = 0$

2.3 $3^{x+1} - 3^{x-1} = 216$

2.4 $2^x = 5^x$

2.5 $3^{2x} - 12.3^x + 27 = 0$

Solutions

2.1
$$2^x = 8$$
$$2^x = 2^3$$
$$x = 3$$

2.2
$$5^{2x+1} - 125^{2x-3} = 0$$
$$5^{2x+1} = 125^{2x-3}$$
$$5^{2x+1} = (5^3)^{2x-3}$$
$$5^{2x+1} = 5^{6x-9}$$
$$2x + 1 = 6x - 9$$
$$10 = 4x$$
$$x = \frac{5}{2}$$

2.3
$$3^{x+1} - 3^{x-1} = 216$$
$$3^x.3^1 - 3^x.3^{-1} = 216$$
$$3^x.(3 - 3^{-1}) = 216$$
$$3^x\left(\frac{8}{3}\right) = 216$$
$$3^x = 216 \div \left(\frac{8}{3}\right)$$
$$3^x = 81$$
$$3^x = 3^4$$
$$\therefore \quad x = 4$$

2.4 $\quad 2^x = 5^x$

$$\frac{2^x}{5^x} = 1$$

$$\left(\frac{2}{5}\right)^x = \left(\frac{2}{5}\right)^0$$

$$x = 0$$

2.5 $\quad 3^{2x} - 12.3^x + 27 = 0$

$$3^x.3^x - 12.3^x + 27 = 0$$

Let $p = 3^x$

$$p.p - 12.p + 27 = 0$$

$$p^2 - 12p + 27 = 0$$

$$(p-3)(p-9) = 0$$

$$p = 3 \text{ or } p = 9$$

$$3^x = 3 \text{ or } 3^x = 3^2$$

$$x = 1 \text{ or } x = 2$$

1.3 Solving equations with surds

To solve equations with surds, arrange the equation with the surd on one side of the equals sign and the other terms on the other side. Square both sides of the equation and solve for the unknown. Remember to CHECK your solutions

WORKED EXAMPLE 3

Solve for x:

3.1 $\sqrt{3x+4} - 5 = 0$

3.2 $\sqrt{3x-5} + 5 = x$

Solutions

3.1 $\quad \sqrt{3x+4} - 5 = 0$

$$\sqrt{3x+4} = 5$$

$$\left(\sqrt{3x+4}\right)^2 = (5)^2$$

$$3x + 4 = 25$$

$$3x = 21$$

$$x = 7$$

3.2 $\sqrt{3x-5}+5=x$

$$\sqrt{3x-5}=x-5$$

$$\left(\sqrt{3x-5}\right)^2=(x-5)^2$$

$$3x-5=x^2-10x+25$$

$$0=x^2-13x+30$$

$$0=(x-3)(x-7)$$

$$x=3 \text{ or } x=7$$

2 QUADRATIC INEQUALITIES

A **quadratic inequality** involves determining the values of x for which the graph of a parabola lies either above or below the x-axis. The following rule is always applicable when working with inequalities, change the direction of the inequality sign whenever you multiply or divide by a negative number.

WORKED EXAMPLE 4

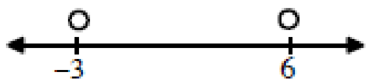
Solve for x

1. $x^2-3x-18>0$
2. $x^2-9<0$
3. $x^2-9\leq 0$

The critical values are: 6 and 3

The inequality reads: **"greater than zero (not equal to)"**

The critical values are not included in the final solution. Therefore, we will use **open dots**.



The next thing to do is to determine the values of x for which the expression on the left is smaller than zero (negative or <0).

To do this one has to test values to the left and right of the critical values.

(a) to the left of -3 (values smaller than -3),

(b) between -3 and 6, and

(c) to the right of 6 (values greater than 6)

(a) Values smaller than -3

Try $x=-4$ $\quad(-4-6)(-4+3)=10$ which is positive. $(+)$

Try $x=-5$ $\quad(-5-6)(-5+3)=22$ which is positive. $(+)$

You may substitute any value less than -3 and the answer always results in a positive number.

(b) Values between -3 and 6

Try $x=-2$ $\quad(-2-6)(-2+3)=-8$ which is negative. $(-)$

Try $x=0$ $\quad(0-6)(0+3)=-18$ which is negative. $(-)$

Try $x=4$ $\quad(4-6)(4+3)=-14$ which is negative. $(-)$

You may substitute any value between -3 and 6 and the answer always results in a negative number.

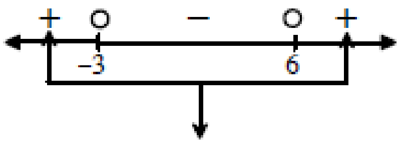
(c) Values greater than 6

Try $x=7$ $\quad(7-6)(7+3)=10$ which is positive. $(+)$

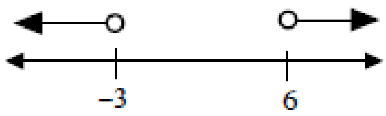
Try $x=8$ $\quad(8-6)(8+3)=22$ which is positive. $(+)$

You may substitute any value greater than 6, and you will see that the answer always results in a positive number.

Let us summarise the results of (a), (b) and (c) on the number line.



Therefore the final solution for $x^2-3x-18>0$ is: $\quad x<-3$ or $x>6$



You may write this in interval notation: $x\in(-\infty;-3)\cup(6;\infty)$

2. $x^2-9<0$

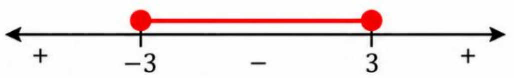
$(x-3)(x+3)<0,\ x\in\mathbb{R}$

| Number line solution

$-3<x<3$	Inequality notation
$x\in(-3,3)$	Set notation
$(-3,3)$	Interval notation

3. $x^2-9\le 0$

$(x-3)(x+3)\le 0,\ x\in\mathbb{R}$



| Number line solution

$-3\le x\le 3$	Inequality notation
$x\in[-3,3]$	Set notation
$[-3,3]$	Interval notation

WORKED EXAMPLE 5

QUESTION 1

1.1 Solve for x:

1.1.1 $x(x-7)=0$ (2)

1.1.2 $x^2-6x+2=0$ (correct to TWO decimal places) (3)

1.1.3 $\sqrt{x-1}+1=x$ (5)

1.1.4 $3^{x+3}-3^{x+2}=486$ (4)

1.2 Given: $f(x)=x^2+3x-4$

1.2.1 Solve for x if $f(x)=0$ (2)

1.2.2 Solve for x if $f(x)<0$ (2)

1.2.3 Determine the values of x for which $f'(x)\geq 0$ (2)

1.3 Solve for x and y: $x=2y$ and $x^2-5xy=-24$ (4)

[24]

WORKED EXAMPLE 6

QUESTION 1

1.1 Solve for x:

1.1.1 $(x-2)(4+x)=0$ (2)

1.1.2 $3x^2-2x=14$ (correct to TWO decimal places) (4)

1.1.3 $2^{x+2}+2^x=20$ (3)

1.2 Solve the following equations simultaneously:

$x=2y+3$

$3x^2-5xy=24+16y$ (6)

1.3 Solve for x: $(x-1)(x-2)<6$ (4)

1.4 The roots of a quadratic equation are: $x=\dfrac{3\pm\sqrt{-k-4}}{2}$

For which values of k are the roots real? (2)

[21]

Solution

QUESTION/*VRAAG* 1

1.1.1	$(x-2)(4+x)=0$ $x=2$ or $x=-4$	✓$x=2$ ✓$x=-4$ (2)
1.1.2	$3x^2-2x-14=0$ $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{2\pm\sqrt{(-2)^2-4(3)(-14)}}{2(3)}$ $=\frac{2\pm\sqrt{172}}{6}$ $x=2{,}52$ or/*of* $x=-1{,}85$	✓ standard form/*standaardvorm* ✓ substitution into correct formula/ *substitusie in korrekte formule* ✓✓ answers/ *antwoorde* (4)
	OR/*OF* $x^2-\frac{2}{3}x+\frac{1}{9}=\frac{14}{3}+\frac{1}{9}$ $\left(x-\frac{1}{3}\right)^2=\frac{43}{9}$ $x-\frac{1}{3}=\pm\frac{\sqrt{43}}{3}$ $\therefore x=\frac{1\pm\sqrt{43}}{3}$ $x=2{,}52$ or/*of* $x=-1{,}85$	✓for adding $\frac{1}{9}$ on both sides/*tel* $\frac{1}{9}$ *by aan beide kante* ✓ $x=\frac{1\pm\sqrt{43}}{3}$ ✓✓answers (4)

1.1.3	$2^{x+2}+2^x=20$ $2^x(2^2+1)=20$ $2^x=\frac{20}{5}$ $2^x=2^2$ $\therefore x=2$	✓common factor/*gemeen. faktor* ✓ simplification/ *vereenvoudiging* ✓ answer/*antwoord* (3)
	OR/*OF* $2^x.2^2+2^x=2^2.5$ $2^x(2^2+1)=2^2.5$ $2^x.5=2^2.5$ $\therefore x=2$	✓ common factor/*gemeen. faktor* ✓ simplification/ *vereenvoudiging* ✓ answer/*antwoord* (3)
	OR/*OF* $4.2^x+2^x=20$ $5.2^x=20$ $2^x=4=2^2$ $\therefore x=2$	✓$5.2^x=20$ ✓$2^x=4$ ✓answer/*antwoord* (3)

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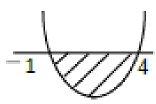
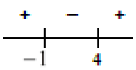
1.2	$x = 2y + 3$ ....(1) $3x^2 - 5xy = 24 + 16y$ ....(2) (1) in (2): $3(2y+3)^2 - 5(2y+3)y = 24 + 16y$ $3(4y^2 + 12y + 9) - 10y^2 - 15y = 24 + 16y$ $12y^2 + 36y + 27 - 10y^2 - 15y - 24 - 16y = 0$ $2y^2 + 5y + 3 = 0$ $(2y+3)(y+1) = 0$ $y = -\frac{3}{2}$ or $y = -1$ $\therefore x = 2\left(-\frac{3}{2}\right) + 3$ *or* $x = 2(-1) + 3$ $x = 0$ *or* $x = 1$ $(0\,;-\frac{3}{2})$ $(1\,;-1)$ **OR/*OF***	✓ substitution/*substitusie* ✓ simplification/ *vereenvoudiging* ✓standard form/ *standaardvorm* ✓factorisation/*faktorisering* ✓*y*-values/*y-waardes* ✓*x*-values/*x-waardes* (6)

	$y = \frac{x-3}{2}$ $3x^2 - 5x\left(\frac{x-3}{2}\right) = 24 + 16\left(\frac{x-3}{2}\right)$ $3x^2 - \frac{5x^2 - 15x}{2} = 24 + \frac{16x - 48}{2}$ $\times 2: 6x^2 - 5x^2 + 15x = 48 + 16x - 48$ $x^2 - x = 0$ $x(x-1) = 0$ $x = 0$ *or* $x = 1$ $y = -\frac{3}{2}$ *or* $y = -1$	✓substitution/*substitusie* ✓simplification/ *vereenvoudiging* ✓standard form / *standard vorm* ✓ factors/*faktore* ✓*x*- values/*x- waardes* ✓ *y*-values/*y*-waardes (6)
1.3	$(x-1)(x-2) < 6$ $x^2 - 3x + 2 < 6$ $x^2 - 3x - 4 < 0$ $(x+1)(x-4) < 0$  **OR/ *OF***  $-1 < x < 4$ or $x \in (-1;4)$	✓ standard form/ *standaardvorm* ✓factorisation/*faktorisering* ✓critical values in the context of inequality / *kritiese waardes in die konteks van die ongelykheid* ✓notation/*notasie* (4)
1.4	$-k - 4 \geq 0$ $k \leq -4$	✓ $-k - 4 \geq 0$ ✓ answer/*antwoord* (2) **[21]**

WORKED EXAMPLE 7

QUESTION 1

1.1 Solve for x:

1.1.1 $x(3x-1)=0$ (2)

1.1.2 $2x^2+13=5x$ (correct to TWO decimal places) (4)

1.1.3 $(x-3)(x+4)\geq 0$ (3)

1.2 Solve for x and y if:

$y = x^2 - 11x + 36$ and $y = 2x - 6$ (5)

Solutions

<u>1.1</u>

<u>1.1.1</u>

$$x(3x-1)=0$$

$$\therefore x=0 \quad \text{or}/\textit{of} \quad x=\frac{1}{3}$$

<u>1.1.2</u>

$$2x^2+13=5x$$

$$2x^2-5x+13=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(2)(13)}}{2(2)}$$

$$x=\text{imaginary or non}-\text{real}$$

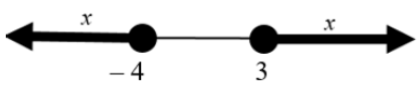
<u>1.1.3</u>

$$(x-3)(x+4)\ \geq 0$$

C.V / *KW* : -4 and / *en* 3

Solution / *Oplossing* $x\ \leq -4$ or / *of* $x\geq 3$

OR/*OF*



<u>1.2</u>

$$y=x^2-11x+36 \quad \text{and}/\textit{en} \quad y=2x-6$$

$$x^2-11x+36=2x-6$$

$$x^2-13x+42=0$$

$$(x-7)(x-6)=0$$

$$x=7 \text{ or}/\textit{of} \ x=6$$

$$y=2(7)-6=8 \quad \text{or}/\textit{of} \ y=2(6)-6=6$$

	OR/*OF* $y = x^2 - 11x + 36$ $\frac{y+6}{2} = x$ $y = \left(\frac{y+6}{2}\right)^2 - 11\left(\frac{y+6}{2}\right) + 36$ $y = \left(\frac{y^2+12y+36}{4}\right) - \left(\frac{11y+66}{2}\right) + 36$ $4y = y^2 + 12y + 36 - 22y - 132 + 144$ $0 = y^2 - 14y + 48$ $(y-8)(y-6) = 0$ $y = 8$ or/*of* $y = 6$ $x = \frac{8+6}{2} = 7$ or/*of* $x = \frac{6+6}{2} = 6$

SECTION 1: ALEGBRA

EXERCISE 1

QUESTION 1

1.1 Solve for x:

1.1.1 $(4x+1)(5-x)=0$ (2)

1.1.2 $x(7x-9)-8=0$ (correct to TWO decimal places) (4)

1.1.3 $-x^2+5x+6>0$ (express the solution in interval notation) (3)

1.2 Given: $y-2x=1$ and $x^2-xy+y^2=7$

1.2.1 Make y the subject of the formula if $y-2x=1$ (1)

1.2.2 Hence, solve for x and y. (5)

EXERCISE 2

QUESTION 1

1.1 Solve for x:

1.1.1 $2x\left(x-\frac{4}{9}\right)=0$ (2)

1.1.2 $6+(2x-5)(x+2)=0$ (correct to TWO decimal places) (4)

1.1.3 $(3-x)(x+2)>0$ (2)

1.2 Given: $y-x+1=0$ and $x^2+xy=3$

1.2.1 Make y the subject of the equation $y-x+1=0$ (1)

1.2.2 Solve for x and y simultaneously. (5)

EXERCISE 3

QUESTION 1

1.1 Given: $f(x)=x(x+2)$

Solve for x if:

1.1.1 $f(x)=0$ (2)

1.1.2 $f(x)\geq 0$ and then represent the solution on a number line. (4)

1.2 Solve for x if $5x^2=2+x$ (rounded off to TWO decimal places) (4)

1.3 Solve algebraically for m and t simultaneously if:
$m-t-1=0$ and $m^2+t^2=5$ (6)

EXERCISE 4

QUESTION 1

1.1 Solve for x:

1.1.1 $x(x + 5) = 0$ (2)

1.1.2 $(x + 1)(x - 3) = 2x$ (Round off your answer to ONE decimal place) (4)

1.1.3 $2x^2 - x - 3 < 0$ (3)

1.2 Solve simultaneously for x and y:

$3y - 2 + x = 0$ and $y^2 - y = xy - x$ (6)

EXERCISES 5

QUESTION 1

1.1 Solve for x:

1.1.1 $x(x + 5) = 0$ (2)

1.1.2 $(x + 1)(x - 3) = 2x$ (Round off your answer to ONE decimal place) (4)

1.1.3 $2x^2 - x - 3 < 0$ (3)

1.2 Solve simultaneously for x and y:

$3y - 2 + x = 0$ and $y^2 - y = xy - x$ (6)

SECTION 1: ANALYTICAL GEOMETRY

1 Introduction

This is previous knowledge to be applied:

- **Straight line graphs:**

The standard form of the straight-line graph **($y = mx + c$),** as well as determining the equation of the line from the information given (sketch or co-ordinates of point/s). And to apply the alternate formula.

$\boldsymbol{y - y1 = m(x - x1)}$ **and** $\boldsymbol{m = \frac{y_2 - y_1}{x_2 - x_1}}$

- **Gradients**

When the gradients of lines are equal, then the lines are parallel.

($m_{AB} = m_{CD}$; $\therefore$ **AB** $\parallel$ **CD)**

If the product of the gradients are equal to $\mathbf{-1}$, then the two lines are perpendicular to each other.

$m_{AB} \times m_{CD} = -1 = \therefore$ **AB** $\perp$ **CD)**

- Angle of Inclination

Another alternative to determine the **gradient** of a line, is to use the **Angle of Inclination**.:

$$\tan\theta = m_{line}$$

- The **Midpoint M** of the line **AB** will be given by;

$$M(\frac{x_A + x_B}{2}; \frac{y_A + y_B}{2})$$

- The **Distance** of the line **AB** between two points **A** and **B** will be given by;

$$d_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

NOTE: If the line **AB** is **Vertical**, then we only consider the $y - values$ of the points, i.e. $d_{AB} = y_{top} - y_{bottom}$

If the line **AB** is **Horizontal**, then we only consider the $x - values$ of the points, $d_{AB} = x_{right} - x_{left}$

Learners needs to learn the following properties of the following quadrilaterals

<table>
<tr><th colspan="2">PROPERTIES OF SPECIAL QUADRILATERALS</th></tr>
<tr><td>

PARALLELOGRAM

- Both pairs of opposite sides are parallel
- Both pairs of opposite sides are equal
- Both pairs of opposite angles are equal
- Diagonals bisect each other

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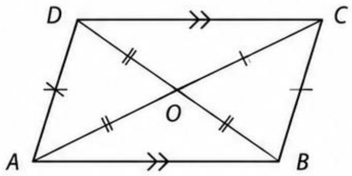

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RECTANGLE

All properties of parallelogram

Plus:

- Both diagonals are equal in length
- All interior angles are equal to 90°

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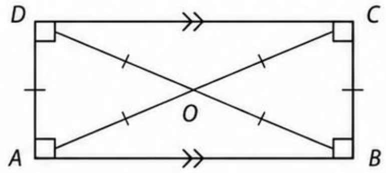

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<tr><td>

RHOMBUS

All properties of parallelogram

Plus:

- All sides are equal
- Diagonals bisect each other perpendicularly
- Diagonals bisect interior angles

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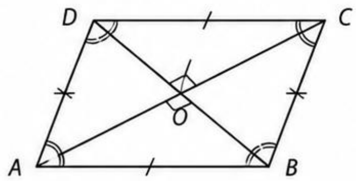

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SQUARE

All properties of a rhombus

Plus:

- All interior angles are 90°
- Diagonals are equal in length

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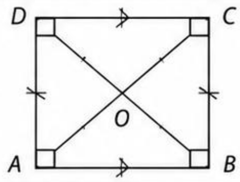

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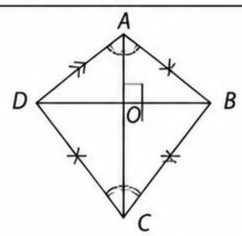
<tr><td>

KITE

- Two pairs of adjacent sides are equal
- Diagonal between equal sides bisects other diagonal
- One pair of opposite angles are equal (unequal sides)
- Diagonal between equal sides bisects interior angles (is axis of symmetry)
- Diagonals intersect perpendicularly

</td><td>



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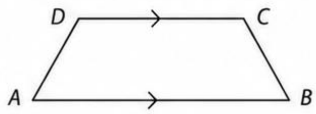
<tr><td>

TRAPEZIUM

- One pair of opposite sides are parallel

</td><td>



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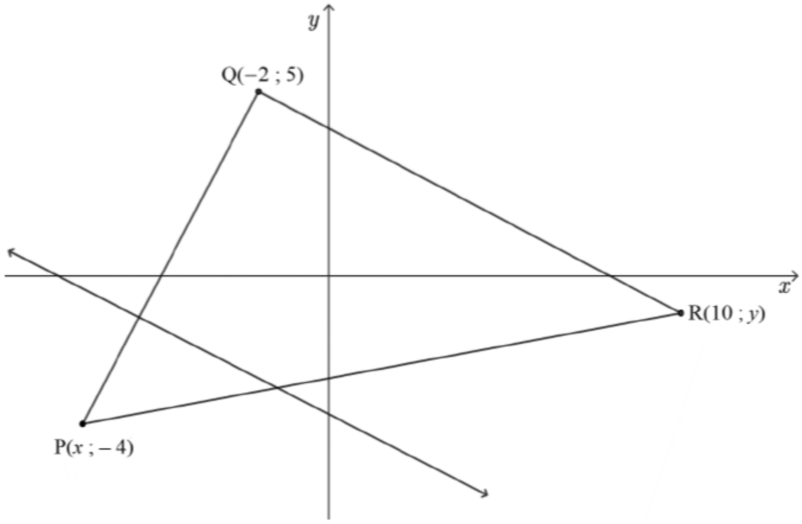
</table>

WORKED EXAMPLE 1

In the diagram below P(x ; –4), Q(–2 ; 5) and R(10 ; y) are points on a cartesian plane. The line $2y + x + 5 = 0$ is drawn such that it is parallel to line segment QR.



1.1 Determine the gradient of QR.

1.2 Show that R is the point (10; −1).

1.3 Calculate the coordinates of M, the midpoint of QR.

1.4 Determine the equation of the line passing through M, which is perpendicular to line QR.

1.5 Given QP = QR.

1.5.1 What type of a triangle is ΔPQR?

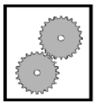
1.5.2 Determine the value of the x-coordinate of P, correct to the nearest whole number.

1.5.3 Hence, calculate the length of PR to the nearest whole number.

Solution

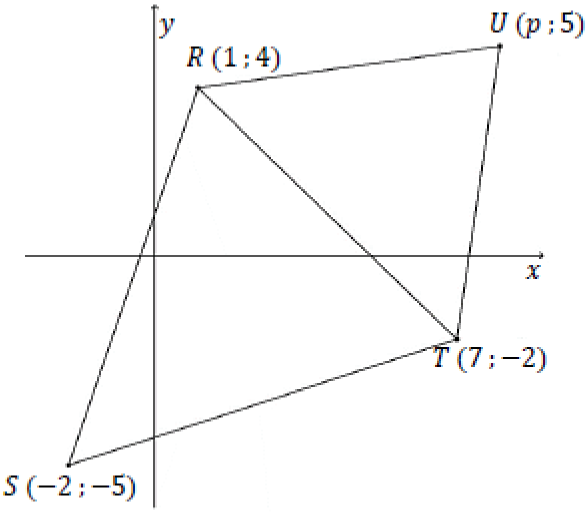
1.1	$2y+x+5=0$ $2y=-x-5$ $y=-\frac{1}{2}x-\frac{5}{2}$ $m_{QR}=-\frac{1}{2}$	✓ S ✓ Ans / *Ant*	**A** **CA**

1.2	$m=\dfrac{y_2-y_1}{x_2-x_1}$ $\dfrac{y-5}{10-(-2)}=-\dfrac{1}{2}$ $2y-10=-12$ $2y=-12+10$ $y=-1$ $\therefore R(10;-1)$	✓ M **A** ✓ S **A** ✓ Ans / *Ant* **CA**
1.3	$M\left(\dfrac{x_1+x_2}{2};\dfrac{y_1+y_2}{2}\right)$ $=M\left(\dfrac{-2+10}{2};\dfrac{5+(-1)}{2}\right)$ $=M(4;2)$	✓ *x*-value / *x-waarde* **A** ✓ *y*-value / *y-waarde* **A**



WORKED EXAMPLE 2

In the diagram below S(− 2 ; − 5), R(1 ; 4), U(pp ; 5) and T(7 ; − 2) are points on a Cartesian plane. The line RT is drawn to join R and T.



2.1 Determine the gradient of RT.

2.2 Calculate the coordinates of Q, the midpoint of RT.

2.3 Determine the length of RT. Leave the answer in the simplest surd form.

2.4 Show that RT⊥SQ.

2.5 Determine the equation of the line passing through S and which is parallel to RT.

2.6 It is given that U is a point, such that the distance of $SU=10\sqrt{2}$. Calculate the value of p.

2.7 Determine whether the line SQ divides ΔSRT in two equal parts. Justify your conclusion with calculations.

Solutions

2.1	$m_{RT}=\dfrac{y_2-y_1}{x_2-x_1}$ $=\dfrac{-2-4}{7-1}$ $=-1$
2.2	$Q\left(\dfrac{x_1+x_2}{2};\dfrac{y_1+y_2}{2}\right)$ $=Q\left(\dfrac{1+7}{2};\dfrac{4+(-2)}{2}\right)$ $=M(4;1)$

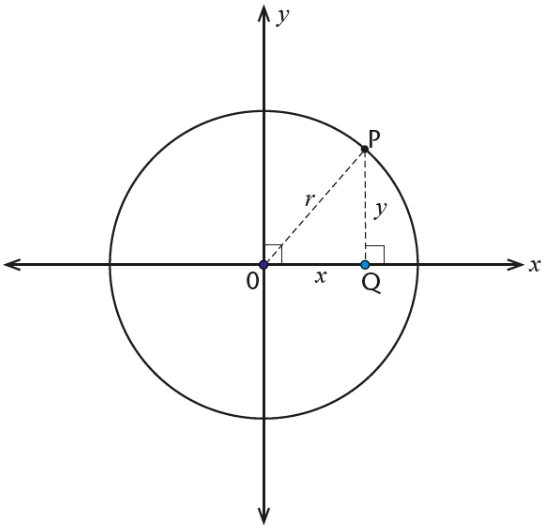
2.3	$RT=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ $RT=\sqrt{(7-1)^2+(-2-4)^2}$ $RT=\sqrt{6^2+(-6)^2}$ $RT=6\sqrt{2}$ *units / eenhede*	✓ SF CA ✓ S CA ✓ RT CA	(3)
2.4	$m_{SQ}=\dfrac{1-(-5)}{4-(-2)}$ $=1$ $m_{RT}\times m_{SQ}=-1\times1$ $=-1$	✓ m_{SQ} CA ✓ Su CA ✓ Prod CA	(3)

2.5	$m = m_{ST} = -1$ $y - y_1 = m(x - x_1)$ $y-(-5) = -1(x-(-2))$ $y = -x - 2 - 5$ $y = -x - 7$	✓ $m = -1$ CA ✓ SF A ✓ S CA ✓ Answer/ *Antwoord* CA	(4)
2.6	$\sqrt{(p-(-2))^2 + (5-(-5))^2} = 10\sqrt{2}$ $\sqrt{(p+2)^2 + (10)^2} = 10\sqrt{2}$ $(p+2)^2 + (10)^2 = (10\sqrt{2})^2$ $(p+2)^2 = 200 - 100$ $p + 2 = \pm\sqrt{100}$ $p = -2 \pm 10$ $p = 8$ and/*en* $p \neq -12$	✓ SF A ✓ Squaring/ *Kwadrateer* CA ✓ Transpose/ *Transponeer* CA ✓ S CA ✓ $p = 8$ CA	(5)

2.7	$SQ = \sqrt{(4+2)^2 + (1+5)^2}$ $SQ = 6\sqrt{2}$ $RQ = 3\sqrt{2}$ $QT = 3\sqrt{2}$ Area/*oppervlakte* $\Delta SQR = \frac{1}{2} RQ \times SQ$ $= \frac{1}{2} \times 3\sqrt{2} \times 6\sqrt{2}$ $= 18 units^2$ / *eenhede*^{2} Area/*oppervlakte* $\Delta SQT = \frac{1}{2} QT \times SQ$ $= \frac{1}{2} \times 3\sqrt{2} \times 6\sqrt{2}$ $= 18 units^2$ / *eenhede*^{2} Area/oppervlakte ΔSQR = Area/oppervlakte ΔSQT $\therefore$ SQ divides ΔSRT into two equal parts/SQ *verdeel* ΔSRT *in twee gelykedele*	✓SF A ✓ SQ CA ✓ SF A ✓ Area/*Oppervlakte* △ SQR CA ✓ SF CA ✓Conclusion/*Slot* CA	(6)
			[25]

2 THE EQUATION OF THE CIRCLE WITH THE CENTRE AT THE ORIGIN

A circle with its centre at origin is defined by the equation $x^2 + y^2 = r^2$, where r represents the radius of the circle.



2.1 Determining the equation of a circle

The equation can be determined when the radius is given or when the coordinates of a point on the circle is given.

WORKED EXAMPLE 2

1 Determine the equation of a circle with its centre at origin and the radius of 3 cm.

2 Determine the equation of the circle with its centre at origin and passes through the point $(\sqrt{7}\ ;\ 4)$

Solutions:

1. $x^2 + y^2 = r^2$

   $x^2 + y^2 = (3)^2$

   $x^2 + y^2 = 9$

2. $x^2 + y^2 = r^2$

   $(\sqrt{7})^2 + (4)^2 = r^2$

   $23 = r^2$

   $\therefore\ x^2 + y^2 = 23$

ACTIVITY 1

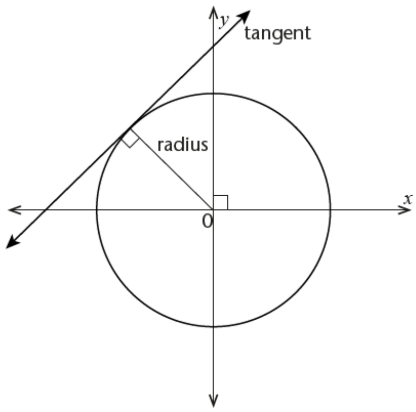
1 Find the equation of the circle with its centre at origin and:

1.1 With radius 11

1.2 Passes through the point $(-2\,;\,-9)$

2.2 THE EQUATION OF A TANGENT

A tangent is a line that touches the circle at only one point. Even if it's extended, it will never touch the circle again.



- Tangent is always perpendicular to the radius at the point of contact. This means the angle between the radius and the tangent is 90^o
- Therefore, $m_{rad} \times m_{tan} = -1 \ (rad \perp tan)$.

WORKED EXAMPLE 3

3.1 Determine the equation of the tangent to the circle defined by $x^2 + y^2 = 2$ at a point $(-1\,;\,1)$.

Solution:

1. $m_{rad} = \frac{y_2 - y_1}{x_2 - x_1}$

$$m_{rad} = \frac{0 - 1}{0 - (-1)}$$

$$m_{rad} = -1$$

$$m_{tan} = \frac{-1}{m_{rad}}$$

$$m_{tan} = \frac{-1}{-1} \qquad \therefore \ m_{tan} = 1$$

$y - y_1 = m(x - x_1)$

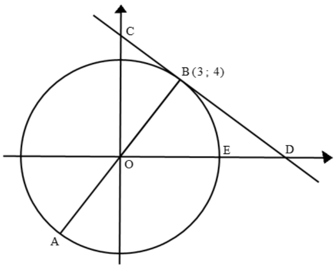
$y - 1 = 1\big(x - (-1)\big)$

$y - 1 = x + 1$

$y = x + 2$

ACTIVITY 2

2.1 The circle below is defined by the equation $x^2 + y^2 = 25$



Determine the equation of the tangent to the circle at B.

2.2 A circle whose centre is at the origin passes through the point $(1; -3)$.

Determine the equation of the tangent to the circle at the point $(1; -3)$.

2.3 POINT OF INTERSECTION

The point of intersection of a circle and a straight line is a point at which the circle and the line meet (where they are equal). We solve equations simultaneously to find the coordinates of the point of intersection.

- A line segment that cuts through the circle (two points of intersection) is called a secant.
- A line segment that touches (one point of intersection) the circle is called a tangent

WORKED EXAMPLE 4

4.1 Determine the point(s) of intersection of the line $x + y = 2$ and the circle $x^2 + y^2 = 2$

Solution:

$x + y = 2$ …………………..…eq 1

$x^2 + y^2 = 2$…………………..… eq 2

From eq 1 $x = 2 - y$ …..…eq 3

Subs eq 3 into eq 2

$(2 - y)^2 + y^2 = 2$

$4 - 4y + y^2 + y^2 - 2 = 0$

$2y^2 - 4y + 2 = 0$

$y^2 - 2y + 1 = 0$

$(y - 1)(y - 1) = 0$

$y = 1$

$x = 1 \qquad (1 ; 1)$

ACTIVITY 3

3.1 Determine whether the line segment $y - x = 1$ will be a secant or a tangent to the circle $x^2 + y^2 = 13$

3.2 Determine the point of tangency if $x - 6y = 37$ is the tangent line to the circle $x^2 + y^2 = 37$

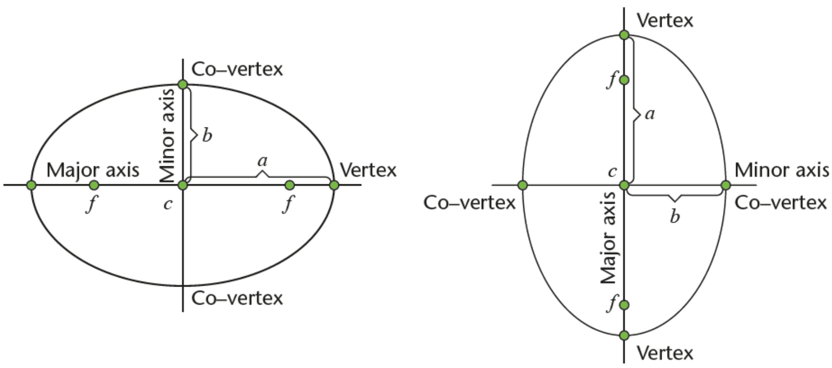
3.3 Determine the point of tangency if $x - 6y = 37$ is the tangent line to the circle $x^2 + y^2 = 37$

3.4 A point $M(p; 2)$ lies on the circle $x^2 + y^2 = 20$ such that the equation of the tangent line at M is $y = -2x + 10$. Determine the value of p.

3 ELLIPSE

The set of points on a cartesian plane, creating an oval shape are called an ellipse. An ellipse can either be vertical or horizontal.

Standard equation of an ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



WORKED EXAMPLE 5

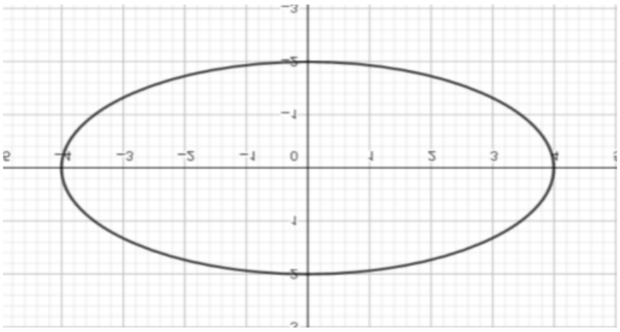
5 Sketch the following ellipse:

5.1 $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$

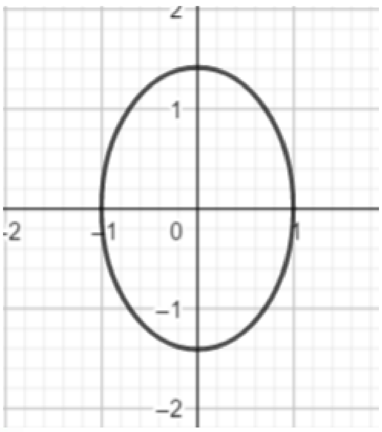
5.2 $\dfrac{x^2}{1} + \dfrac{y^2}{2} = 1$

Solutions:

5.1



5.2



ACTIVITY 4

1. Sketch the following ellipses:

   1.1. $\frac{x^2}{9} + \frac{y^2}{25} = 1$

   1.2. $\frac{x^2}{6} + \frac{y^2}{12} = 1$

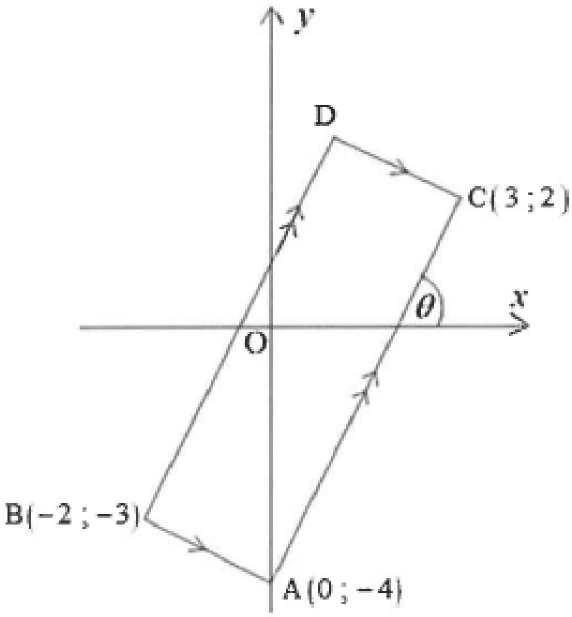
EXERCISES

1 Introduction

Exercise 1

QUESTION 1.

The diagram below shows parallelogram ABDC with vertices $A(0;-4)$, $B(-2;-3)$, D and $C(3;2)$.
AB || CD and AC || BD.
The angle of inclination of AC with the positive x-axis is θ.



1.1 Write down the length of OA. (1)

1.2 Determine the midpoint of AB. (2)

1.3 Determine the gradient of AC. (2)

1.4 Hence, determine the size of angle θ. (2)

1.5 Complete the following statement:

If two lines are parallel, then their gradients are … (1)

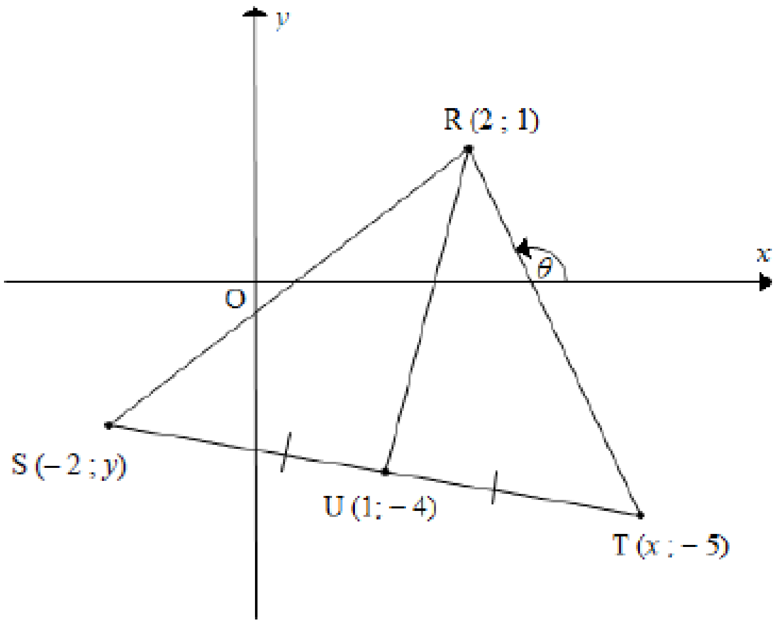
1.6 Hence, determine the equation of BD in the form $y = ...$ (3)

1.7 Determine the gradient of a line that passes through point B and has an inclination of α, where $\cos\alpha = -\frac{\sqrt{2}}{2}$ (3)

Exercise 2

QUESTION 1

The diagram below shows ΔRST with vertices $R(2\,;\,1)$, $S(-2\,;\,y)$ and $T(x\,;\,-5)$.
The angle of inclination of RT with the positive x-axis is θ.
$U(1\,;\,-4)$ is the midpoint of ST.
RU is drawn.



Determine:

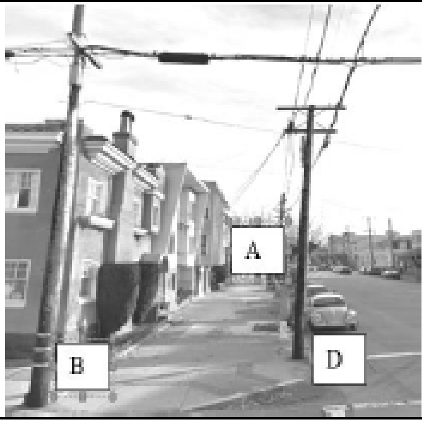
1.1	The length of RU (leave the answer in surd form)	(2)
1.2	The values of x and y	(2)
1.3	The gradient of RT	(2)
1.4	The size of θ	(3)
1.5	The equation of the circle with centre O, passing through point S	(2)
		[11]

Exercise 3

QUESTION 1

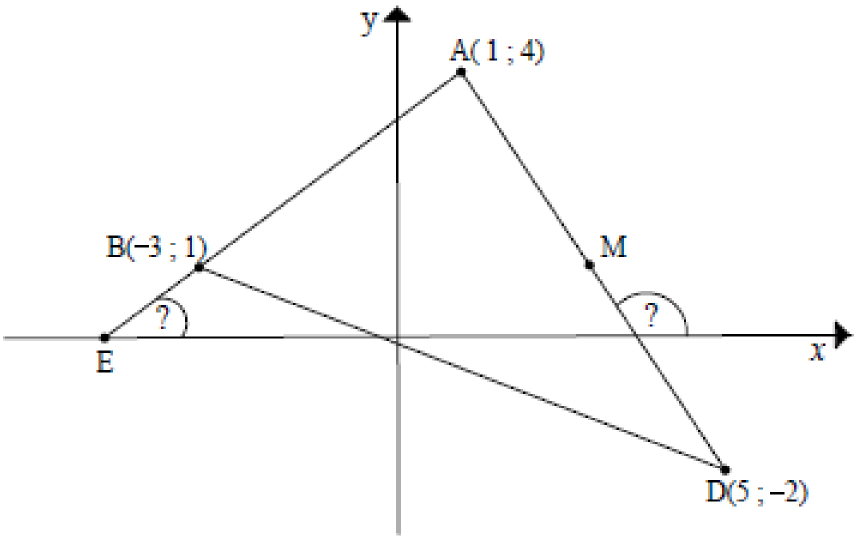
The picture shows electric poles and wires.

Assume that an electrician is standing next to electric pole A. Two assistants, to the right and left of the electrician respectively, are standing next to electric poles B and D.



The diagram below, NOT drawn to scale, models the above situation in a Cartesian plane.

ΔABD has vertices $A(1;4)$, $B(-3;1)$ and $D(5;-2)$. The angle formed by the x-axis and AE is α and the angle formed by the x-axis and AD is β.



Determine:

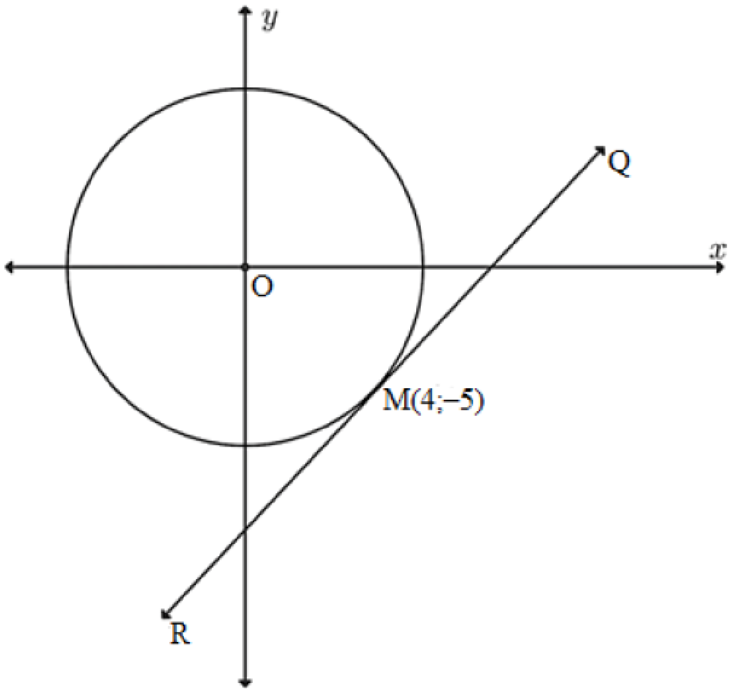
1.1	The length of AD (leave your answer in simplified surd form)	(3)
1.2	The coordinates of M, the midpoint of AD	(2)
1.3	The equation of straight line MC (in the form $ax+by+c=0$) if MC $\parallel$ AB.	(5)
1.4	The size of α.	(2)
1.5	The size of $B\hat{A}D$.	(4)
		[16]

2 THE EQUATION OF THE CIRCLE WITH THE CENTRE AT THE ORIGIN

EXERCISE 1

QUESTION 2

2.1 In the diagram below, $O(0\ ;\ 0)$ is the centre of the circle defined by $x^2 + y^2 = r^2$. QR is a straight line that intersect the circle at $M(4;-5)$.



2.1.1 What is the name given to line QR? (1)

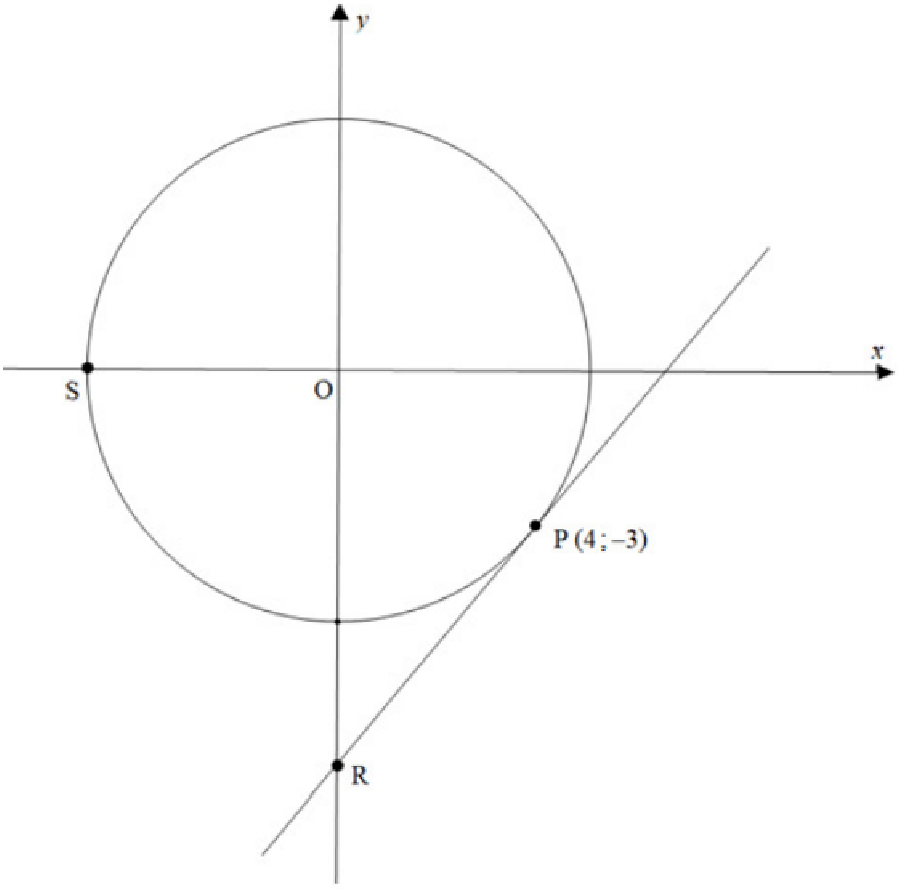
2.1.2 Determine the equation of the circle. (2)

2.1.3 Determine the equation of line QR. (4)

EXERCISE 2

QUESTION 2

2.1 In the diagram below, O is the centre of the circle defined by $x^2 + y^2 = r^2$
PR is a tangent to the circle at point P.
Point S is an x-intercept of the circle.
Point R is the y-intercept of line PR.

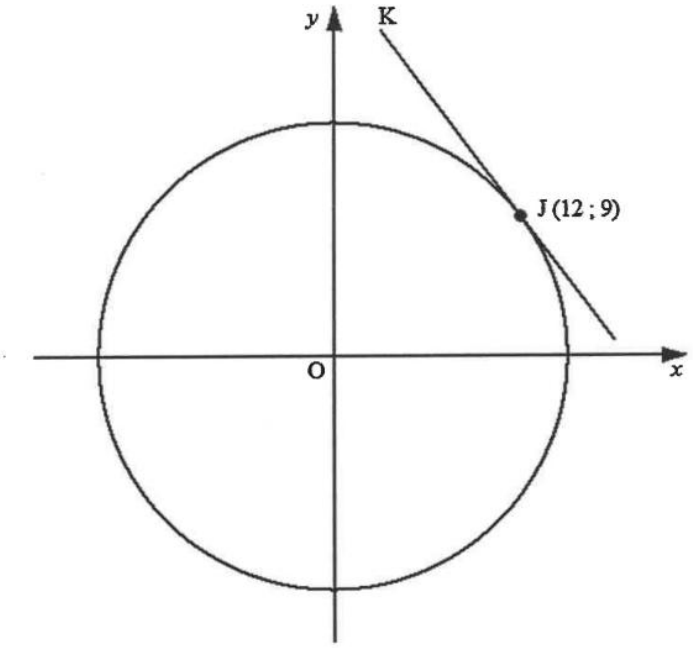


2.1.1	Determine the equation of the circle.	(2)
2.1.2	Write down the coordinates of S.	(2)
2.1.3	Determine the equation of the tangent in the form $y = \ldots$	(4)
2.1.4	Write down the y-coordinate of point R.	(1)

EXERCISE 3

QUESTION 2

2.1 In the diagram below, O is the centre of the circle. JK is a tangent to the circle at point $J(12\ ;\ 9)$.



2.1.1 Determine the equation of the circle passing through J. (2)

2.1.2 Complete the following:

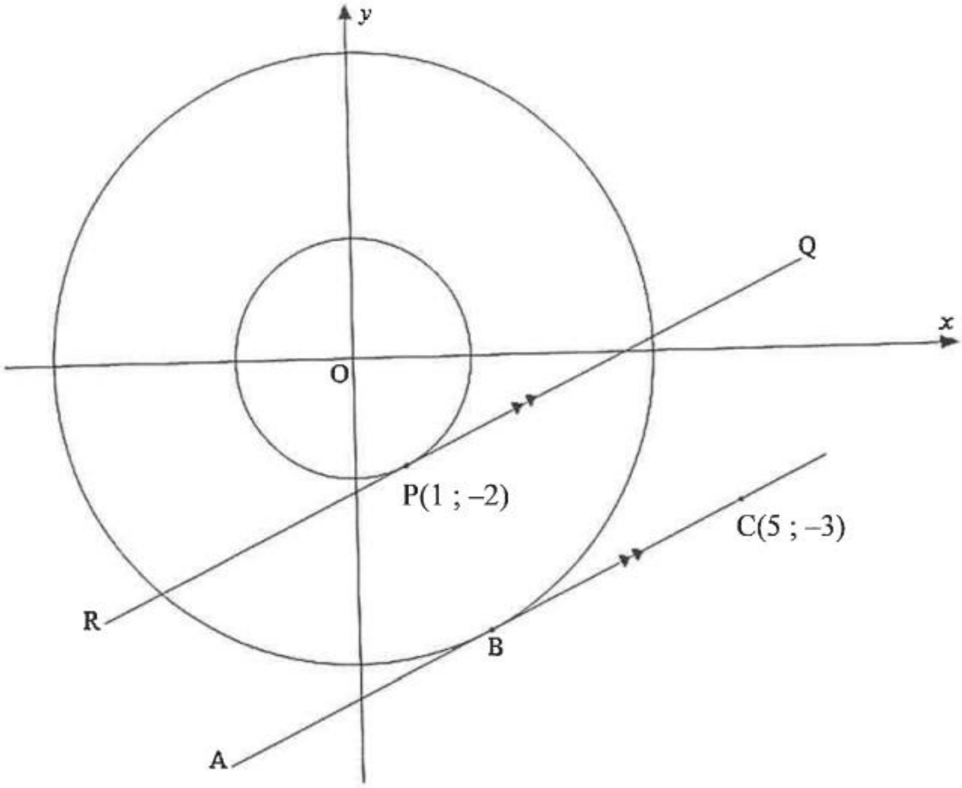
$m_{OJ} \times m_{JK} = \ldots$ (1)

2.1.3 Determine the equation of JK in the form $y = \ldots$ (4)

EXERCISE 4

QUESTION 2

2.1 In the diagram below, O is the centre of both the smaller and the larger circles.
RQ is a tangent to the smaller circle at point $P(1\ ;\ -2)$.
AC is a tangent to the larger circle at point B with $C(5\ ;\ -3)$.
$RQ \parallel AC$



2.1.1	Determine the equation of the smaller circle.	(2)
2.1.2	Write down the gradient of OP.	(1)
2.1.3	Give a reason why OP is perpendicular to RQ.	(1)
2.1.4	Hence, determine the gradient of AC.	(2)
2.1.5	Hence, determine the equation of AC in the form $y = \ldots$	(3)

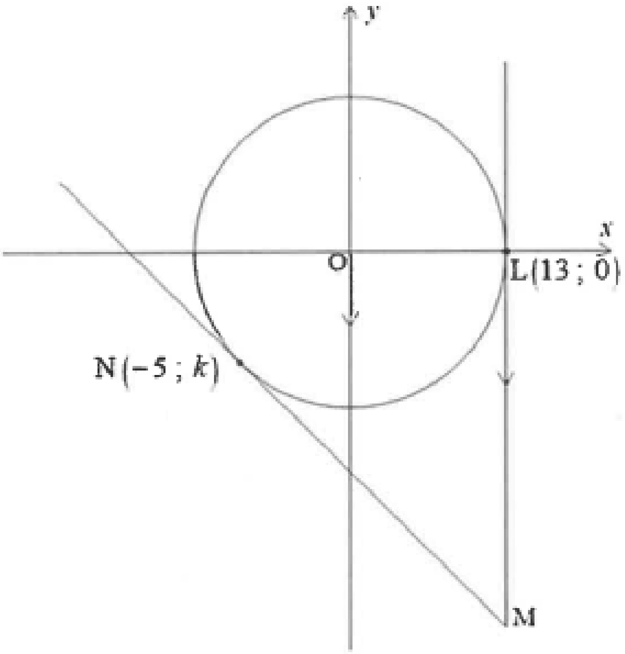
EXERCISE 5

QUESTION 2

2.1 In the diagram below, O is the centre of the circle which passes through points $L(13;0)$ and $N(-5;k)$.

Tangents MN and ML touch the circle at points N and L respectively.

ML $\parallel$ y-axis.



2.1.1 Determine the equation of the circle. (2)

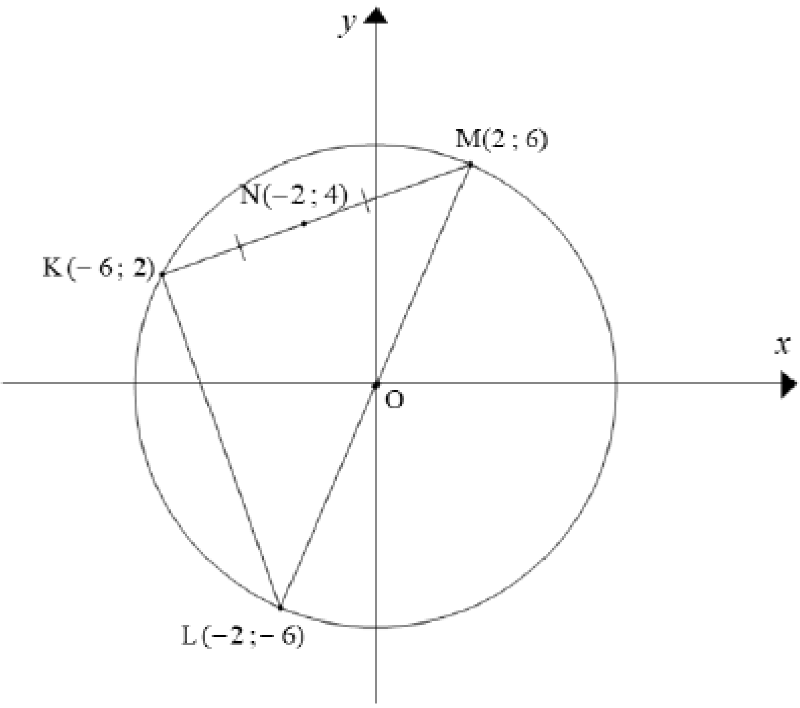
2.1.2 Determine the numerical value of k. (2)

2.1.3 Hence, determine the coordinates of point M, the point of intersection of the two tangents. (5)

EXERCISE 6

QUESTION 2

2.1 In the diagram below, O is the centre of the circle KLM with vertices K(– 6 ; 2), L(–2 ; – 6) and M(2 ; 6), as shown.
Point N(– 2 ; 4) is the midpoint of KM.



2.1.1 Determine the equation of the line ML. (3)

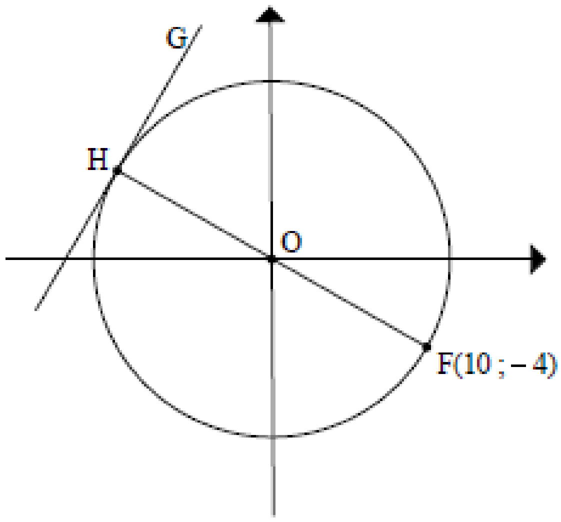
2.1.2 Show, using **analytical geometry methods**, that $KL \perp KM$. (4)

2.1.3 Show that $ON = \frac{1}{2}KL$. (3)

EXERCISE 7

QUESTION 2

2.1 In the diagram below, $O(0;0)$ is the centre of the circle and $F(10;-4)$ is a point on the circle. FH is the diameter of the circle and GH is a tangent to the circle at H.



Determine:

2.1.1 The length of the radius of the circle (2)

2.1.2 The gradient of OH (2)

2.1.3 The equation of GH (4)

3 ELLIPSE

EXERCISE 1

2.2 Given the equation: $\frac{x^2}{32}+\frac{y^2}{81}=1$

2.2.1 Express the equation in the form: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (1)

2.2.2 Hence sketch the graph defined by $\frac{x^2}{32}+\frac{y^2}{81}=1$ on the grid provided, in the ANSWER BOOK. Clearly show ALL the intercepts with the axes. (3)

EXERCISE 2

2.2 Given: $\frac{x^2}{1}+\frac{y^2}{9}=1$

2.2.1 Express the equation in the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (1)

2.2.2 Hence, sketch the graph of the ellipse. (2)

EXERCISE 3

2.2 Given: $\frac{x^2}{11}+\frac{y^2}{64}=1$

2.2.1 Express the equation in the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (1)

2.2.2 Hence, sketch the graph defined by $\frac{x^2}{11}+\frac{y^2}{64}=1$ (2)

EXERCISE 4

2.2 Given: $\frac{x^2}{36}+\frac{y^2}{16}=1$

2.2.1 Express the equation in the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (1)

2.2.2 Hence, sketch the graph defined by $\frac{x^2}{36}+\frac{y^2}{16}=1$ (2)

EXERCISE 5

2.2 Sketch the graph defined by: $\frac{x^2}{25}+\frac{y^2}{4}=1$ (3)

EXERCISE 6

2.2 Sketch the graph of the ellipse defined by $\frac{x^2}{16}+\frac{y^2}{7}=1$. (3)

EXERCISE 7

2.2 Sketch the graph defined by:

$\frac{x^2}{16}+\frac{y^2}{10}=1$

Clearly show ALL the intercepts with the axes. (3)

SECTION 2: Euclidean Geometry

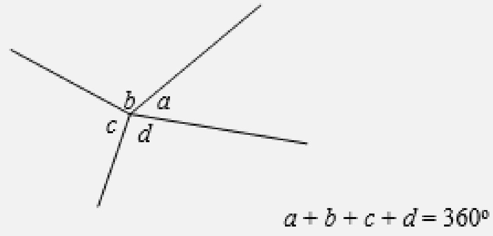
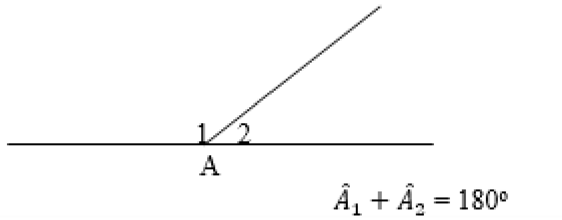
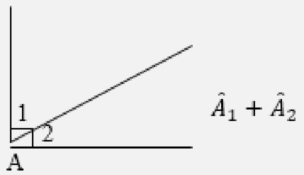
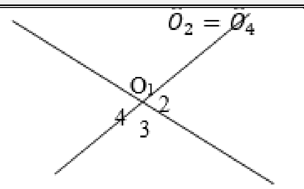
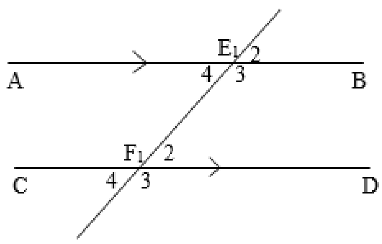
1. Acceptable reasons

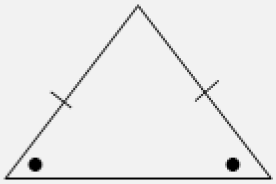
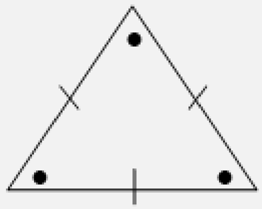
THEOREM STATEMENT	ACCEPTABLE REASON(S)
LINES	
The adjacent angles on a straight line are supplementary.	∠s on a str line
If the adjacent angles are supplementary, the outer arms of these angles form a straight line.	adj ∠s supp
The adjacent angles in a revolution add up to 360°.	∠s round a pt **OR** ∠s in a rev
Vertically opposite angles are equal.	vert opp ∠s =
If AB ∥ CD, then the alternate angles are equal.	alt ∠s; AB ∥ CD
If AB ∥ CD, then the corresponding angles are equal.	corresp ∠s; AB ∥ CD
If AB ∥ CD, then the co-interior angles are supplementary.	co-int ∠s; AB ∥ CD
If the alternate angles between two lines are equal, then the lines are parallel.	alt ∠s =
If the corresponding angles between two lines are equal, then the lines are parallel.	corresp ∠s =
If the co-interior angles between two lines are supplementary, then the lines are parallel.	co-int ∠s supp
TRIANGLES	
The interior angles of a triangle are supplementary.	∠ sum in Δ **OR** sum of ∠s in Δ **OR** Int ∠s Δ
The exterior angle of a triangle is equal to the sum of the interior opposite angles.	ext ∠ of Δ
The angles opposite the equal sides in an isosceles triangle are equal.	∠s opp equal sides
The sides opposite the equal angles in an isosceles triangle are equal.	sides opp equal ∠s
In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.	Pythagoras **OR** Theorem of Pythagoras
If the square of the longest side in a triangle is equal to the sum of the squares of the other two sides then the triangle is right-angled.	Converse Pythagoras **OR** Converse Theorem of Pythagoras
If three sides of one triangle are respectively equal to three sides of another triangle, the triangles are congruent.	SSS
If two sides and an included angle of one triangle are respectively equal to two sides and an included angle of another triangle, the triangles are congruent.	SAS **OR** S∠S
If two angles and one side of one triangle are respectively equal to two angles and the corresponding side in another triangle, the triangles are congruent.	AAS **OR** ∠∠S
If in two right-angled triangles, the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of the other, the triangles are congruent	RHS **OR** 90°HS
The line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side	Midpt Theorem

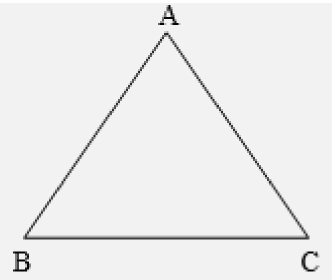
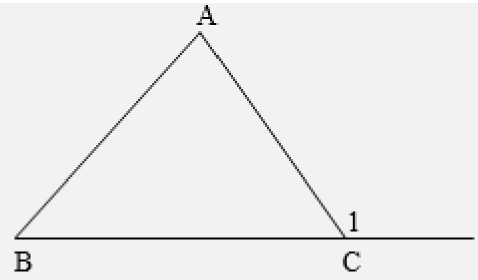
THEOREM STATEMENT	ACCEPTABLE REASON(S)
The line drawn from the midpoint of one side of a triangle, parallel to another side, bisects the third side.	line through midpt || to 2^{nd} side
A line drawn parallel to one side of a triangle divides the other two sides proportionally.	line || one side of Δ **OR** prop theorem; name || lines
If a line divides two sides of a triangle in the same proportion, then the line is parallel to the third side.	line divides two sides of Δ in prop
If two triangles are equiangular, then the corresponding sides are in proportion (and consequently the triangles are similar).	||| Δs **OR** equiangular Δs
If the corresponding sides of two triangles are proportional, then the triangles are equiangular (and consequently the triangles are similar).	Sides of Δ in prop
If triangles (or parallelograms) are on the same base (or on bases of equal length) and between the same parallel lines, then the triangles (or parallelograms) have equal areas.	same base; same height **OR** equal bases; equal height
CIRCLES	
The tangent to a circle is perpendicular to the radius/diameter of the circle at the point of contact.	tan ⊥ radius tan ⊥ diameter
If a line is drawn perpendicular to a radius/diameter at the point where the radius/diameter meets the circle, then the line is a tangent to the circle.	line ⊥ radius **OR** converse tan ⊥ radius **OR** converse tan ⊥ diameter
The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord.	line from centre to midpt of chord
The line drawn from the centre of a circle perpendicular to a chord bisects the chord.	line from centre ⊥ to chord
The perpendicular bisector of a chord passes through the centre of the circle;	perp bisector of chord
The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)	∠ at centre = 2 ×∠ at circumference
The angle subtended by the diameter at the circumference of the circle is 90°.	∠s in semi- circle **OR** diameter subtends right angle
If the angle subtended by a chord at the circumference of the circle is 90°, then the chord is a diameter.	chord subtends 90° **OR** converse ∠s in semi -circle
Angles subtended by a chord of the circle, on the same side of the chord, are equal	∠s in the same seg.
If a line segment joining two points subtends equal angles at two points on the same side of the line segment, then the four points are concyclic.	line subtends equal ∠s **OR** converse ∠s in the same seg.
Equal chords subtend equal angles at the circumference of the circle.	equal chords; equal ∠s
Equal chords subtend equal angles at the centre of the circle.	equal chords; equal ∠s

THEOREM STATEMENT	ACCEPTABLE REASON(S)
Equal chords in equal circles subtend equal angles at the centre of the circles.	equal circles; equal chords; equal $\angle$s
The opposite angles of a cyclic quadrilateral are supplementary	opp $\angle$s of cyclic quad
If the opposite angles of a quadrilateral are supplementary then the quadrilateral is cyclic.	opp $\angle$s quad supp **OR** converse opp $\angle$s of cyclic quad
The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.	ext $\angle$ of cyclic quad
If the exterior angle of a quadrilateral is equal to the interior opposite angle of the quadrilateral, then the quadrilateral is cyclic.	ext $\angle$ = int opp $\angle$ **OR** converse ext $\angle$ of cyclic quad
Two tangents drawn to a circle from the same point outside the circle are equal in length	Tans from common pt **OR** Tans from same pt
The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.	tan chord theorem
If a line is drawn through the end-point of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.	converse tan chord theorem **OR** $\angle$ between line and chord
QUADRILATERALS	
The interior angles of a quadrilateral add up to 360°.	sum of $\angle$s in quad
The opposite sides of a parallelogram are parallel.	opp sides of ||m
If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.	opp sides of quad are ||
The opposite sides of a parallelogram are equal in length.	opp sides of ||m
If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.	opp sides of quad are = **OR** converse opp sides of a parm
The opposite angles of a parallelogram are equal.	opp $\angle$s of ||m
If the opposite angles of a quadrilateral are equal then the quadrilateral is a parallelogram.	opp $\angle$s of quad are = **OR** converse opp angles of a parm
The diagonals of a parallelogram bisect each other.	diag of ||m
If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.	diags of quad bisect each other **OR** converse diags of a parm
If one pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.	pair of opp sides = and ||
The diagonals of a parallelogram bisect its area.	diag bisect area of ||m
The diagonals of a rhombus bisect at right angles.	diags of rhombus
The diagonals of a rhombus bisect the interior angles.	diags of rhombus
All four sides of a rhombus are equal in length.	sides of rhombus
All four sides of a square are equal in length.	sides of square
The diagonals of a rectangle are equal in length.	diags of rect
The diagonals of a kite intersect at right-angles.	diags of kite
A diagonal of a kite bisects the other diagonal.	diag of kite
A diagonal of a kite bisects the opposite angles	diag of kite

2. REVISION OF BASIC RESULTS ESTABLISHED IN EARLIER GRADES

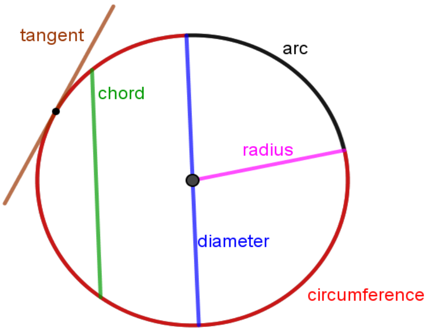
<table>
<tr><th colspan="2">REVISION OF BASIC RESULTS ESTABLISHED IN EARLIER GRADES</th></tr>
<tr><th colspan="2">Intersecting lines</th></tr>
<tr><td>1.1 Angles around a point add up to 360°.
(Revolution)</td><td>
</td></tr>
<tr><td>1.2 The sum of adjacent angles on a straight line is 180°. Two angles that add up to 180° are said to be supplementary.</td><td>
</td></tr>
<tr><td>1.3 Two angles are complementary if their sum is 90° or a right angle.</td><td>
</td></tr>
<tr><td>1.4 Whenever two straight lines intersect, the vertically opposite angles are equal in size.</td><td>
</td></tr>
<tr><th colspan="2">Parallel lines</th></tr>
<tr><td>➢ Alternate angles are equal
$\hat{E}_3 = \hat{F}_1$ and $\hat{E}_4 = \hat{F}_2$
➢ Corresponding angles are equal
$\hat{E}_1 = \hat{F}_1$; $\hat{E}_2 = \hat{F}_2$; $\hat{E}_3 = \hat{F}_3$ and $\hat{E}_4 = \hat{F}_4$
➢ Co-interior angles are supplementary
$\hat{E}_4 + \hat{F}_1 = 180^o$ and $\hat{E}_3 + \hat{F}_2 = 180^o$</td><td>
</td></tr>
</table>

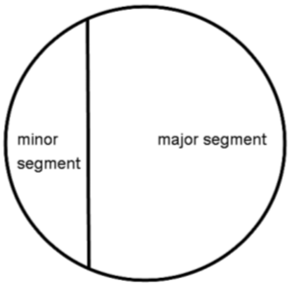
Types of triangles	
Isosceles triangle	**Equilateral triangle**
➢ Exactly two sides are equal in length ➢ The angles opposite equal sides are equal. 	➢ All three sides are equal in length ➢ All three angles are equal, i.e. each angles is equal to 60°. 

Some general properties of triangles	
❖ The sum of interior angles in a triangle is 180°.  $\hat{A} + \hat{B} + \hat{C} = 180^o$	❖ The exterior angle of a triangle is equal to the sum of the interior opposite angles  $\hat{C}_1 = \hat{A} + \hat{B}$

SECTION 3 : Circle geometry

Different parts of a circle:





Circle theorems are divided into four groups :

CENTER THEOREMS	TANGENT THEOREMS	CYCLIC QUAD THEORMES	EQUAL CHORDS COROLLARY
1. The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle. 2. The angle subtended by the diameter at the circumference of the circle is 90 3. The line drawn from the Center of the circle perpendicular to the chord bisects the chord.	1. The angle between the tangent to a circle and a chord drawn from the point of contact is equal to the angle in the alternate segment. 2. Two Tangents drawn to a circle from the same point outside the circle are equal in length . 3. The tangent to a circle is perpendicular to the radius/diameter of the circle at the point of contact.	1. The opposite angles of a cyclic quadrilateral are supplementary. 2. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. 3. Angles subtended by a chord/segment of a circle, on the same side of the chord/segment are equal.	1. Equal chords subtend equal angles at the circumference of the circle. 2. Equal chords subtend equal angles at the center of the circle.

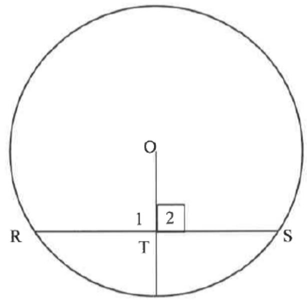
Classical method: This is done by writing two columns (HOLY-CROSS), the first being a list of statements and the second column a matching list of legal justifications. i.e. theorems and axioms referred to as REASONS.

STATEMENT	**REASON**
1. $\hat{A}_2 = \hat{C}_1 = 35°$	Alt $\angle' s$, $AB \parallel CD$
2.$M\hat{A}P + 35° = 75°$	Ext $\angle \; of \; \Delta$

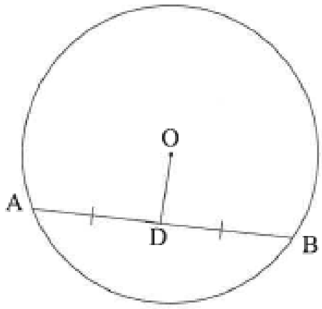
PROOFS

Five proofs of theorems that are examinable from grade 11.

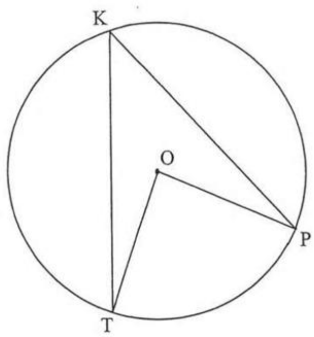
A. In the diagram below, O is the centre of the circle and point T lies on chord RS. Prove the theorem which states that if $OT \perp RTS$ then RT=TS.



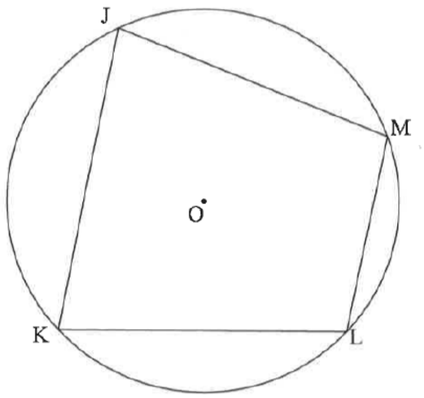
B. In the diagram, O is the centre of a circle. OD bisects AB. Prove the theorem that states that the line from the centre of a circle that bisects a chord is perpendicular to the chord, i.e. $OD \perp AB$.



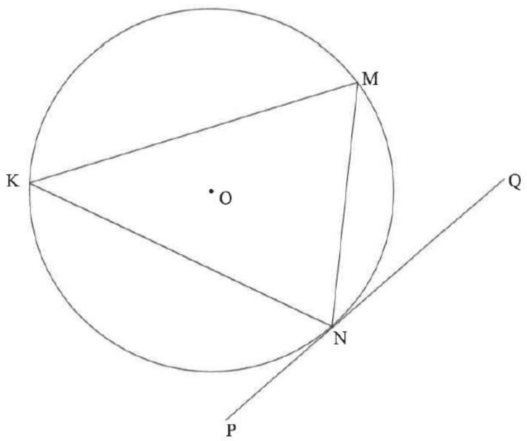
C. In the diagram below, O is the centre of the circle. Use the diagram to prove the theorem which states prove that $T\hat{O}P = 2T\hat{K}P$.



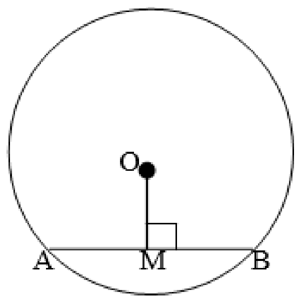
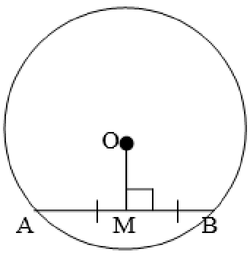
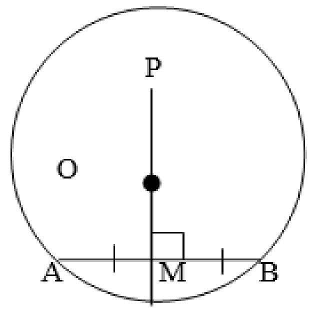
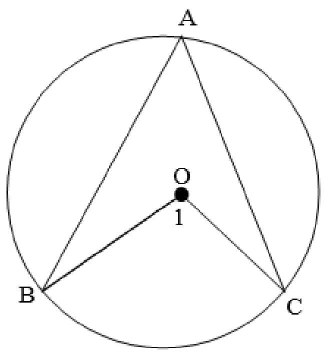
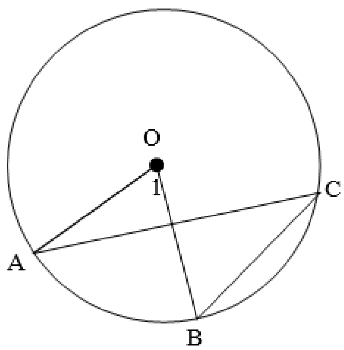
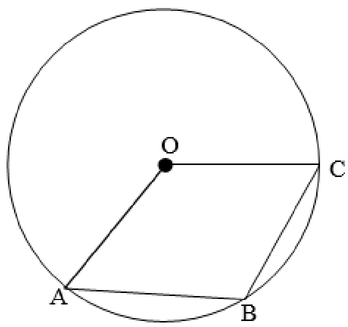
D. In the diagram, JKLM is a cyclic quadrilateral and the circle has centre O. Prove the theorem which states that $\hat{J} + \hat{L} = 180°$



E. In the diagram below, chords KM, MN are drawn in the circle with centre O. PNQ is the tangent to the circle at N. Prove that $M\hat{N}Q = \hat{K}$

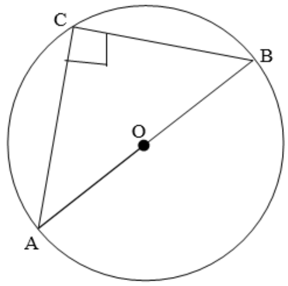


1. Center theorems

<table>
<tr>
<td>The line drawn from centre of the circle perpendicular to a chord bisects the chord.
<u>(line from centre ⊥ to chord)</u>

If OM ⊥ AB, then
AM = MB</td>
<td>The line segment joining the centre of the circle to the midpoint of a chord is perpendicular to the chord.
<u>(line from centre to midpoint of chord)</u>

If AM = MB, then
OM ⊥ AB</td>
<td>The perpendicular bisector of a chord passes through the centre of the circle.
<u>(perp bisector of chord)</u>

If OP ⊥ AB and AM = MB, then PM passes through O.</td>
</tr>
<tr>
<td colspan="3">B.
The angle which an arc of a circle subtends at the center of the circle is twice the angle it subtends at the circumference of the circle. <u>(∠at centre = 2 × ∠ at circle)</u>
  
$\hat{O}_1 = 2\hat{A}$ $\hat{O}_1 = 2\hat{C}$ Reflex $A\hat{O}C = 2\hat{B}$</td>
</tr>
</table>

C. The angle subtended at the circle by the diameter is a right angle.

(∠s in semi-circle) OR **(diameter subtends right anlge)**

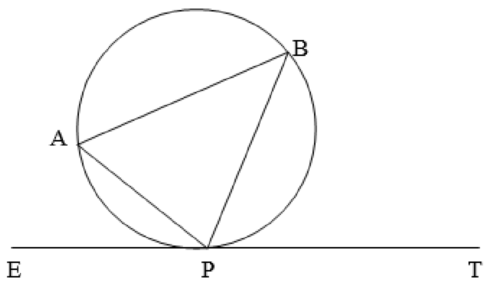


$\hat{C} = 90°$ (AB is a diameter)

2. **Tangents theorems**

A. The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.

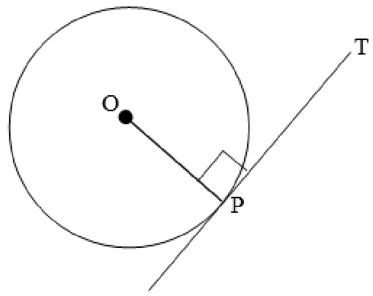
(tan-chord theorem)



$B\hat{P}T = \hat{A}$ and $A\hat{P}E = \hat{B}$

B. A tangent to a circle is perpendicular to the radius at the point of contact.

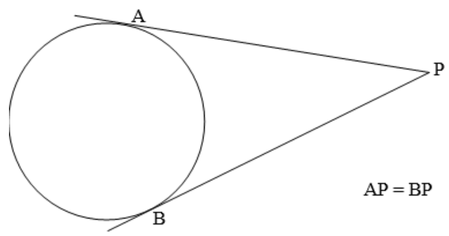
(tan ⊥ radius)



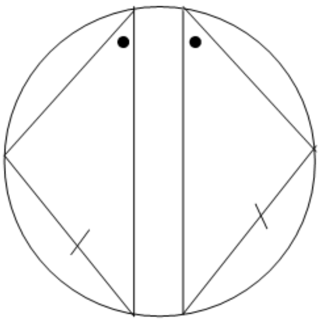
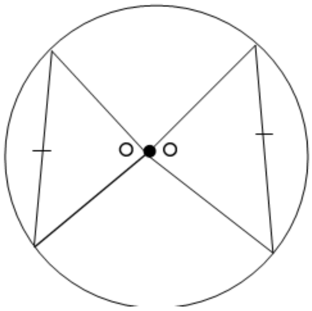
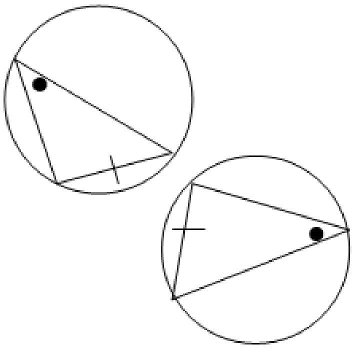
PT ⊥ OP

Two tangents drawn to a circle from the same point outside the circle are equal in length

(tan. from same pt.)



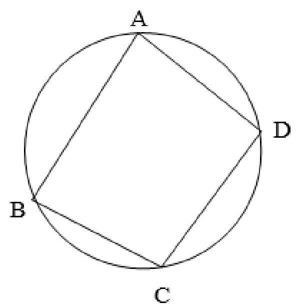
3. Equal chords corollaries

Equal chords subtend equal angles at the circumference of the circle.	Equal chords subtend equal angles at the centre of the circle	Equal chords, in equal circles subtend equal angles.
		

4. Cyclic quad theorems

A. The opposite angles of a cyclic quadrilateral are supplementary.

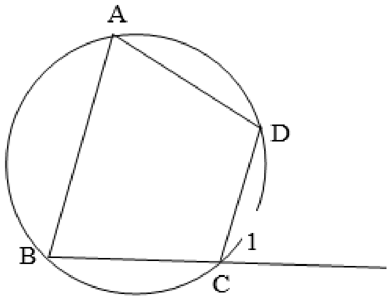
<u>(opp ∠'s of cyclic quad)</u>



$\hat{A} + \hat{C} = 180°$ and $\hat{B} + \hat{D} = 180°$

B. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

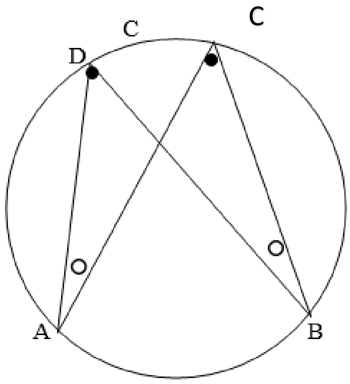
<u>(ext. ∠ of cyclic quad.)</u>



$$\hat{C}_1 = \hat{A}$$

C. Angles subtended by a chord of the circle, on the same side of the chord, are equal.

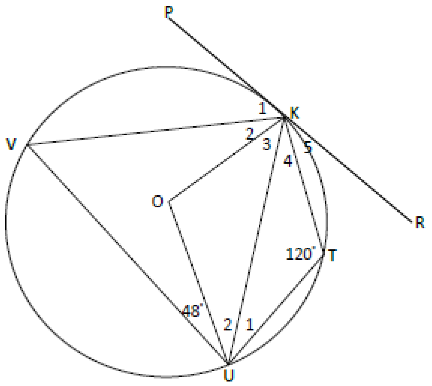
<u>(∠'s in the same seg)</u>



$$\widehat{D} = \hat{C} \text{ and } \hat{A} = \widehat{B}$$

WORKED EXAMPLE 1

In the diagram below, O is the centre of the circle KTUV. PKR is a tangent to the circle at K.
$O\hat{U}V = 48^\circ$ and $K\hat{T}U = 120^\circ$



2.1 Use the above information to fill in the missing information for (a) to (e).

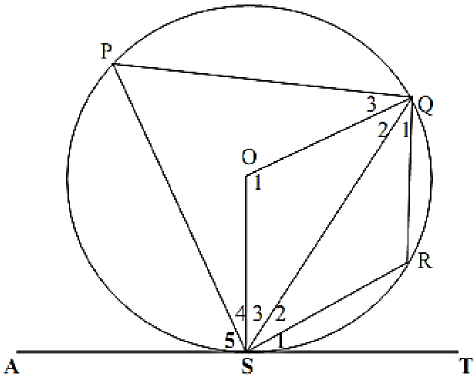
STATEMENT	REASON
$\hat{V} + 120^\circ = 180^\circ$ $\therefore \hat{V} = 60^\circ$	(a)
$K\hat{O}U = 120^\circ$	(b)
$\hat{U}_2 =$ (c)	$OU = OK$ radii
$\hat{K}_1 = 78^\circ$	(d)
$\hat{K}_2 = 12^\circ$	(e)

Solutions

2.1	(a) Opposite angles of a cyclic quad are supplementary.
	(b) Angle at centre is twice angle at circumference.
	(c) $\hat{U} = 30^\circ$
	(d) Tan chord theorem
	(e) Radius is perpendicular to tangent

WORKED EXAMPLE 2

1.3 In the diagram below, AST is a tangent to a circle O at S.
$R\hat{S}T = \hat{S}_1 = 23°$ and QR = RS.



Calculate, with reasons, the sizes of:

1.3.1 $Q\hat{S}R$

1.3.2 $\hat{R}$

1.3.3 $\hat{P}$

1.3.4 $\hat{O}_1$

Solution

1.3.1 $\hat{S}_1 = \hat{Q}_1 = 23°$ …(angle between tangent and chord)

$\therefore\ \hat{S}_2 = \hat{Q}_1 = 23°$ …(RS = RQ)

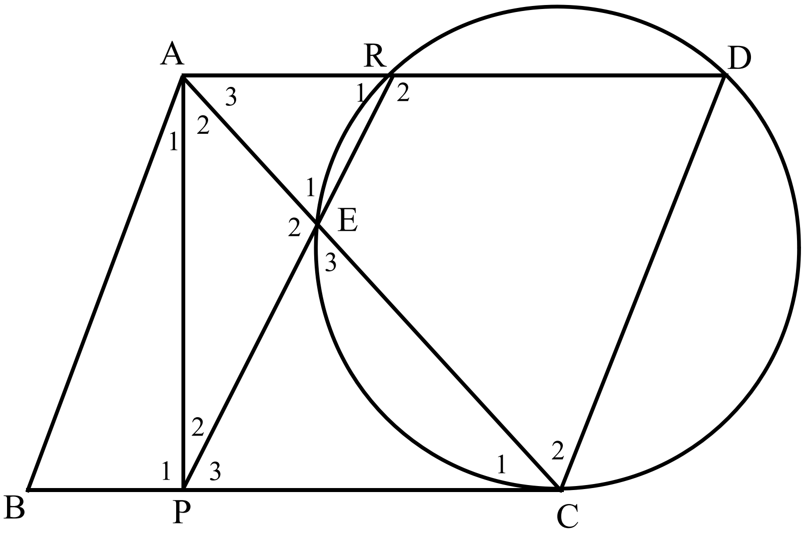
1.3.2 $\therefore\ \hat{R} = 180° - (23° + 23°)$
$= 134°$….(angle sum of triangle)

1.3.3 $\hat{P} = 180° - 134°$
$= 46°$ ……… (Opposite angles of a cyclic quad are supp.)

1.3.4 $\hat{O}_1 = 2\hat{P} = 92°$ …. (angle at the centre is twice…)

WORKED EXAMPLE 3

In the diagram below, ABCD is a parallelogram and RDCE is a cyclic quadrilateral. Diagonal AC cuts the circle in E and RE is produced to cut BC in P. AP is joined.



3.1 Prove that $\hat{R}_1 = \hat{A}_1 + \hat{A}_2$. (3)

3.2 Prove that ABPE is a cyclic quadrilateral. (4)

Solutions

3.1	$\hat{R}_1 = \hat{C}_2$ $\hat{C}_2 = \hat{A}_1 + \hat{A}_2$ $\therefore \hat{R}_1 = \hat{A}_1 + \hat{A}_2$	ext $\angle$ of a cyclic quad alt $\angle s =$; AB || CD
3.2	$\hat{R}_1 = \hat{P}_3$ But $\hat{R}_1 = \hat{A}_1 + \hat{A}_2$ $\therefore \hat{P}_3 = \hat{A}_1 + \hat{A}_2$ $\therefore$ ABPE is a cyclic quad	alt $\angle s =$; AD || BC proved ext $\angle =$ int opp $\angle$

SECTION 3: Euclidean Geometry

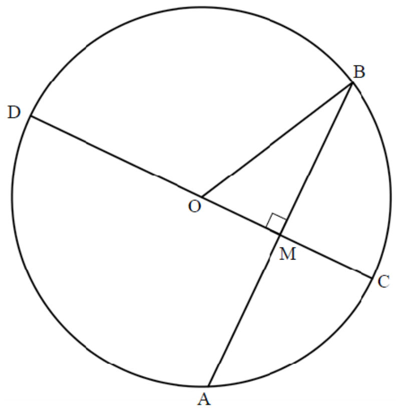
EXERCISES

Exercise 1

Question 10

In the diagram below, O is the centre of the circle. Chord AB is perpendicular to diameter DC.

$CM : MD = 4 : 9$ and $AB = 24$ units

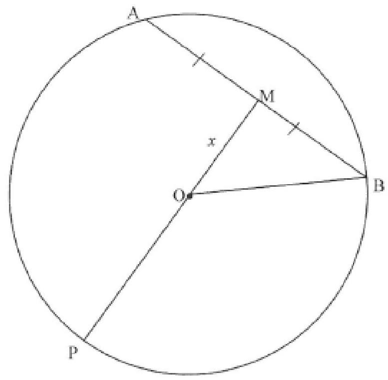


10.1 Determine an expression for DC in terms of x if $CM = 4x$ units	(1)
10.2 Determine an expression for OM in terms of x.	(2)
10.3 Hence, or otherwise, calculate the length of the radius	(4)

Exercise 2

Question 7

In the diagram, AB is a chord of the circle with centre O, M is the midpoint of AB, MO is produced to P, where P is a point on the circle. $OM = x$ units, $AB = 20$ units and $\frac{PM}{OM} = \frac{5}{2}$.



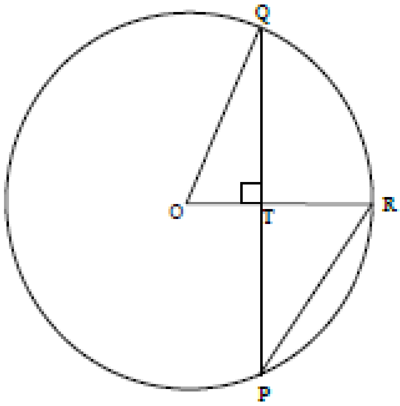
7.1 Write down the length of MB.	(1)
7.2 Give a reason why OM ⊥ AB.	(1)
7.3 Show that $OP = \frac{3x}{2}$ units.	(2)

7.4 Calculate the value of x.	(3)

Exercise 3

Question 20

In the diagram below, PQ is the chord of circle O. OR perpendicular to PQ and OR intersects PQ at T. If the radius of the circle is 13 cm and PT = 12 cm.



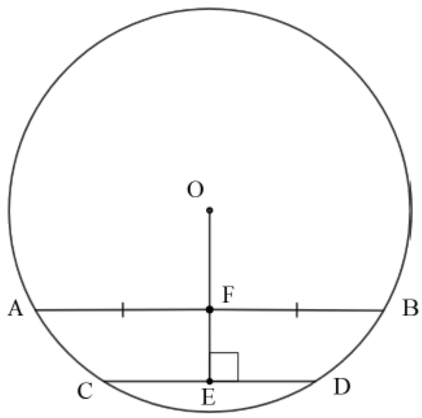
Calculate the length of: 20.1 PQ	(2)
20.2 PR	(4)

Exercise 4

Question 6

In the diagram below AB = 80 cm, CD = 48 cm and OF is 10 cm less than OE.

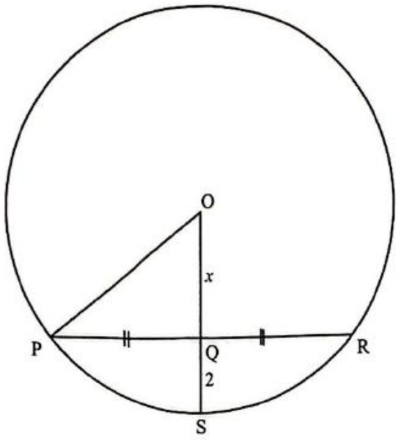
OF = x cm, AF = FB and OE ⊥ CD.



6.1 Determine the following lengths in terms of x: 6.1.1 OE	(2)
6.1.2 OC	(3)
6.2 Determine the numerical value of x.	(5)

Exercise 5

8.2 In the diagram below, O is the centre of the circle. Q is the midpoint of chord PR. OQS is the radius of the circle. PR = 8 units, OQ = x units and QS = 2 units.



8.2.1 Determine, giving reasons, the size of $O\hat{Q}P$.	(2)
8.2.2 Calculate the length of PO.	(5)

Exercise 6

Question 7

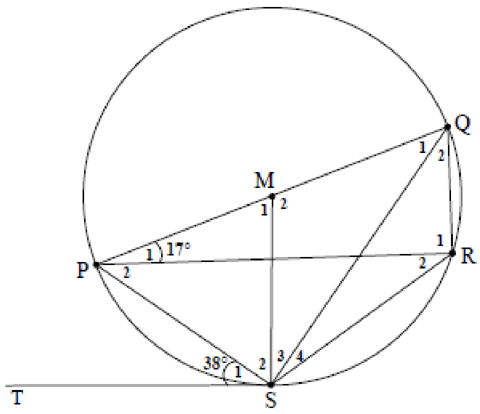
7.1 Complete the following theorem statement :

The angle between the tangent to a circle and the chord drawn from the point of contact is equal to ….. (1)

7.2 In the diagram below, PQ is the diameter of circle PQRS with centre M.

TS is the tangent to the circle at point S.

$\widehat{S_1} = 38°$ and $\widehat{P_1} = 17°$



Determine, giving reasons, the sizes of :

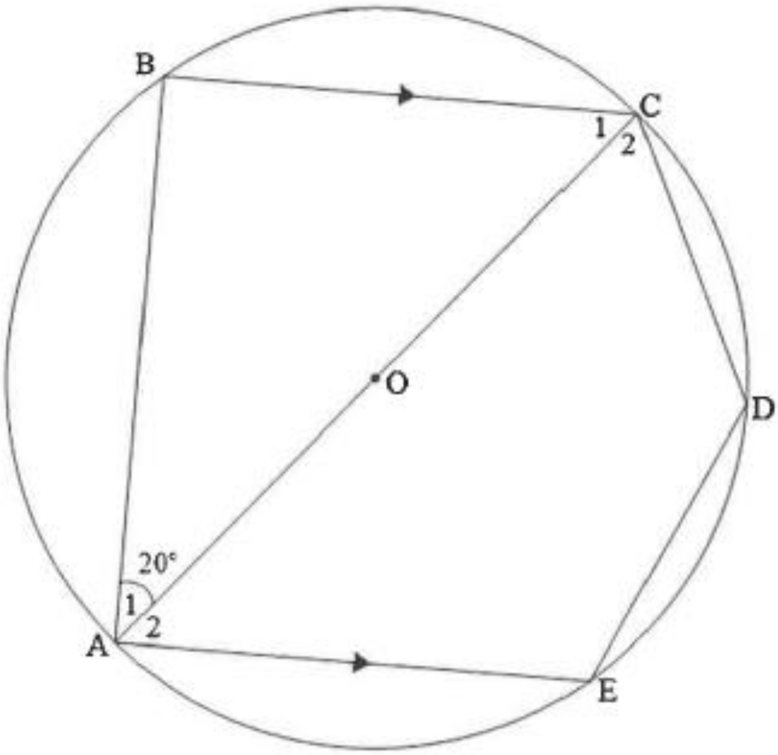
7.2.1 $\widehat{R_2}$	(2)

7.2.2 $\widehat{M_1}$	(2)
7.2.3 $\widehat{S_2}$	(2)
7.2.4 $\widehat{Q_2}$	(5)
7.2.5 Give a reason why PM is not parallel to SR	(1)

Exercise 7

QUESTION 7

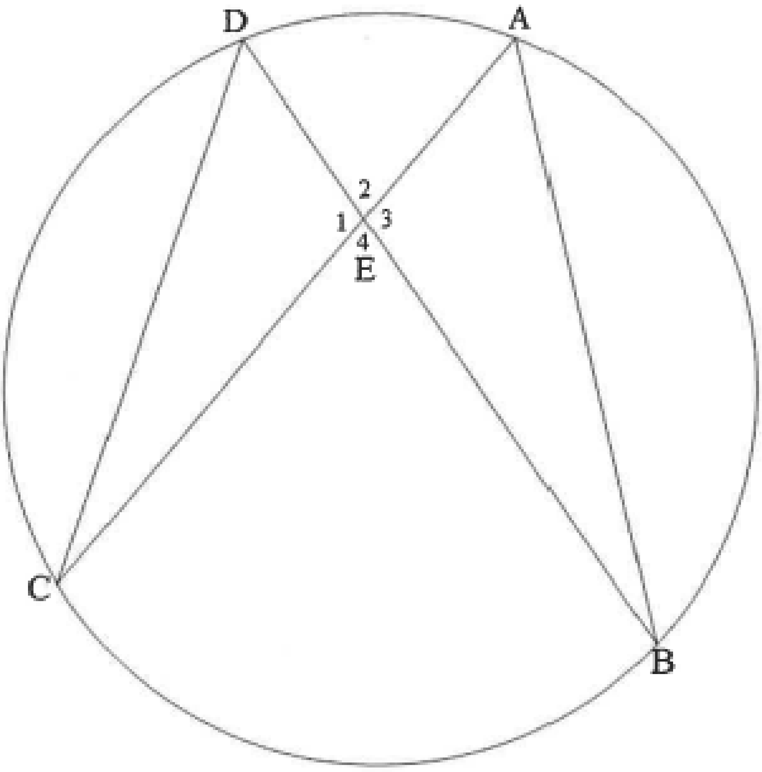
7.1 Below is drawn a circle with centre O.
A, B, C, D and E are on the circle.
CA is a diameter.
AB, CD and DE are drawn.
BC || AE
$\hat{A}_1 = 20°$



7.1.1 Give a reason why $\hat{B} = 90°$. (1)

7.1.2 Determine, with reasons, the size of $\hat{D}$. (3)

7.2 In the diagram below, A, B, C and D are four points on the circumference of the circle.
Chords BD and AC intersect at point E.
Chords DC and AB are drawn.



7.2.1 Complete the statement of the following theorem:

Angles subtended by a chord of a circle, on the same side of the chord, are … (1)

7.2.2 Write reasons for the statements given below:

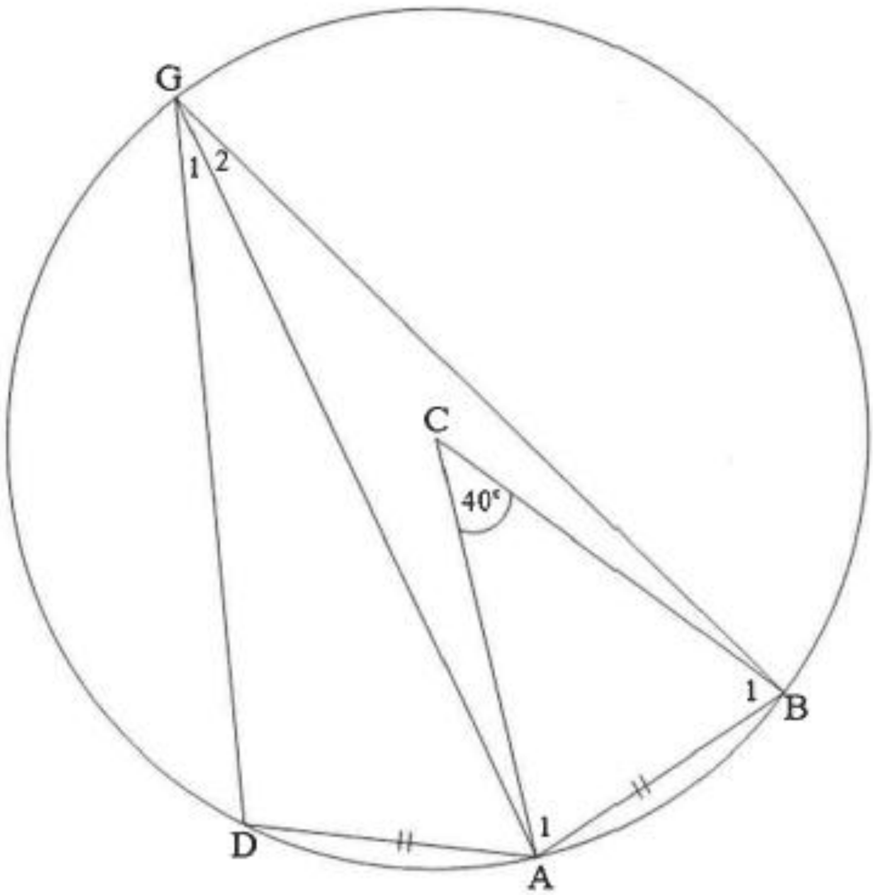
STATEMENT	REASON
(a) $C\hat{D}B = C\hat{A}B$	…
(b) $\hat{E}_1 = \hat{E}_3$	…
(c) $\Delta DEC \| \Delta AEB$	…

(1)
(1)
(1)
[8]

Exercise 8

QUESTION 8

8.1 In the diagram below, G, B, A and D are points on a circle with centre C. GD, GA and GB are drawn.
$AB = AD$
$A\hat{C}B = 40°$

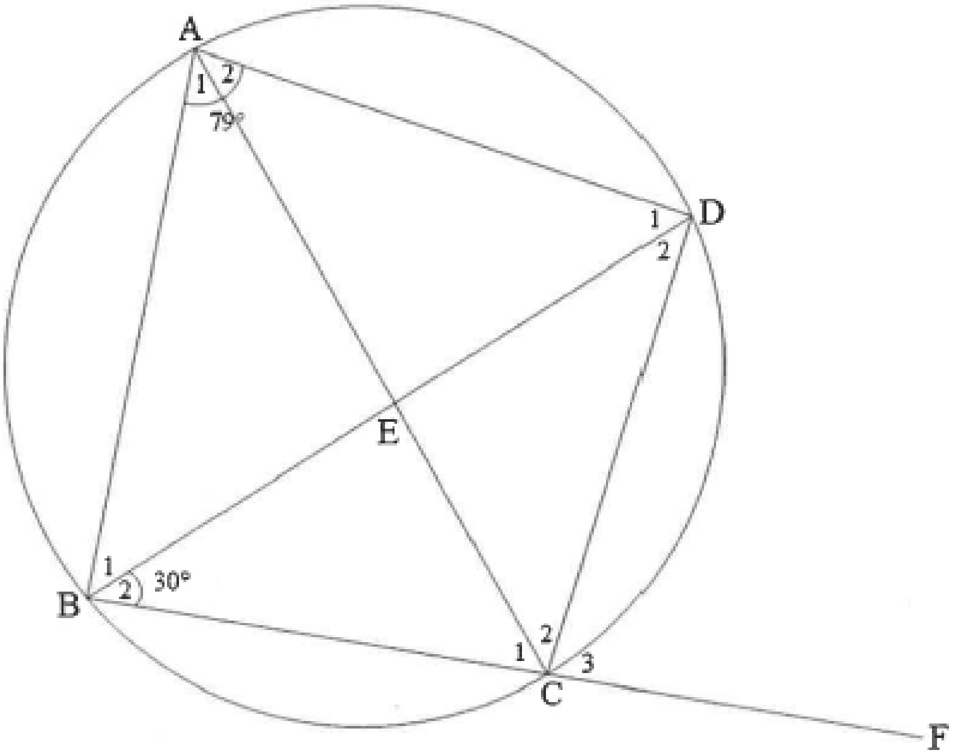


8.1.1	Write down TWO angles that will be equal if $AB = AD$.	(1)
8.1.2	Determine, with a reason, the size of $\hat{G}_2$.	(2)
8.1.3	Give a reason why $\hat{A}_1 = \hat{B}_1$.	(1)
8.1.4	Determine, with a reason, the size of $\hat{A}_1$.	(2)
8.1.5	Determine, with a reason, the size of $D\hat{A}C$.	(3)

8.2 Complete the statement of the following theorem by filling in the missing information:

The (8.2.1) … angle of a cyclic quadrilateral is equal to the (8.2.2) … opposite angle. (2)

8.3 In the diagram below, A, B, C and D lie on the circumference of the circle.
E is the point of intersection of chords BD and AC.
Chord BC is produced to F.
$B\hat{A}D = 79°$ and $\hat{B}_2 = 30°$



8.3.1 Determine, with reasons, the size of the following angles:

(a) $\hat{C}_3$ (2)

(b) $\hat{D}_2$ (2)

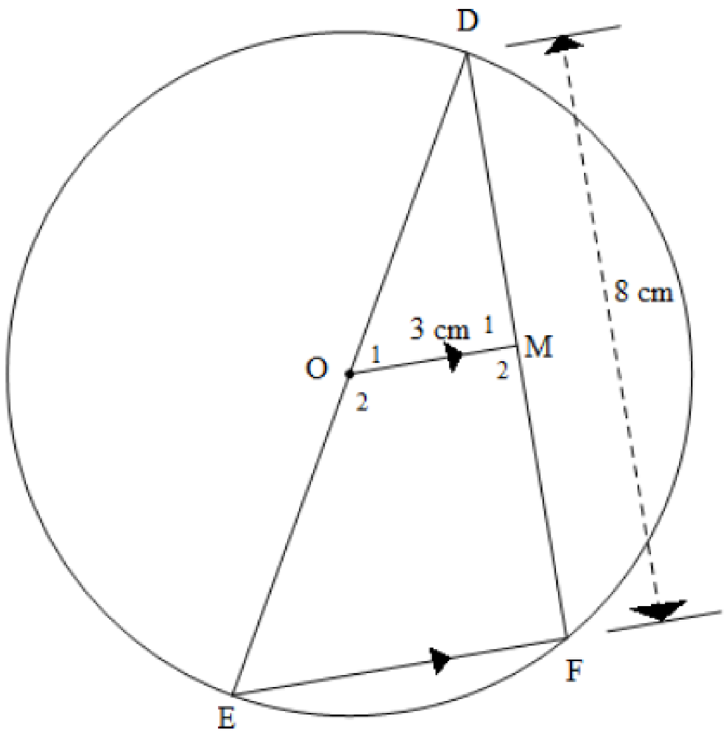
(c) $\hat{D}_1$ if it is given that $AB \parallel CD$ (3)

8.3.2 Show that $CE = DE$ (4)
[22]

Exercise 9

QUESTION 7

In the diagram below, O is the centre of the circle DEF.
DE is a diameter and M is a point on DF such that OM || EF.
DF = 8 cm and OM = 3 cm.

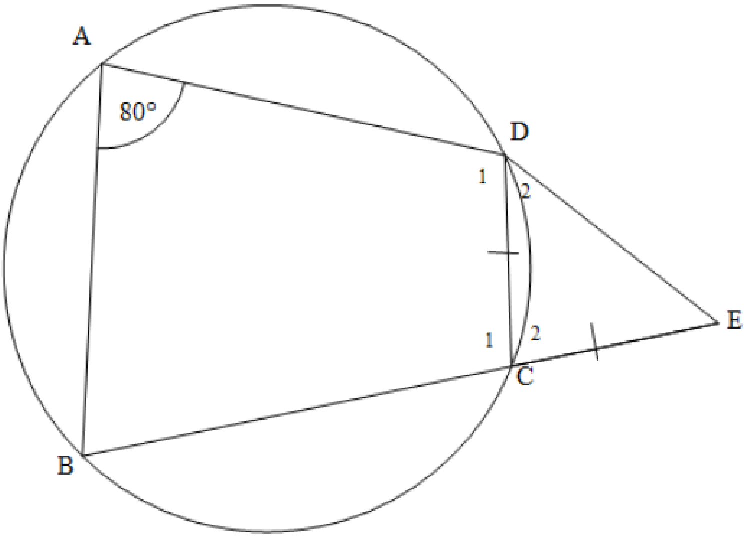


7.1	Give a reason why $\hat{F} = 90°$.	(1)
7.2	Name, giving a reason, another right angle in the diagram.	(2)
7.3	Give a reason why DM = MF.	(1)
7.4	Calculate, giving reasons, the length of the diameter of the circle.	(4)
		[8]

Exercise 10

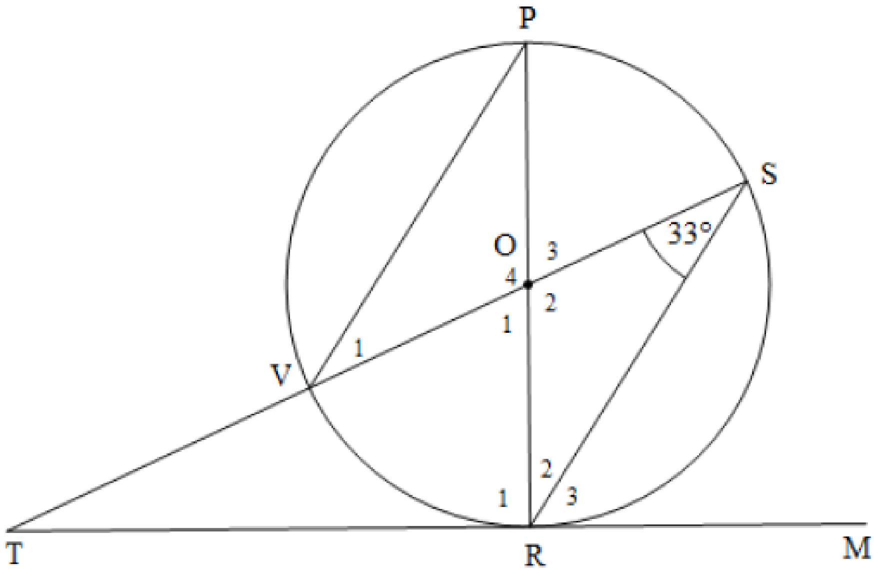
QUESTION 8

8.1 In the diagram below, ABCD is a cyclic quadrilateral.
BC is extended to meet DE at E such that DC = CE.
$\hat{A} = 80°$



Determine, giving reasons, the size of $\hat{E}$. (6)

8.2 In the diagram below, O is the centre of circle PVRS.
PR and SV are diameters.
The tangent MR to the circle at R meets SV produced at T.
$R\hat{S}V = 33°$



8.2.1 Calculate, giving reasons, the sizes of the following angles:

(a) $\hat{P}$ (2)

(b) $\hat{O}_1$ (2)

(c) $\hat{T}$ (3)

8.2.2 Show that PV || SR. (3)

[16]

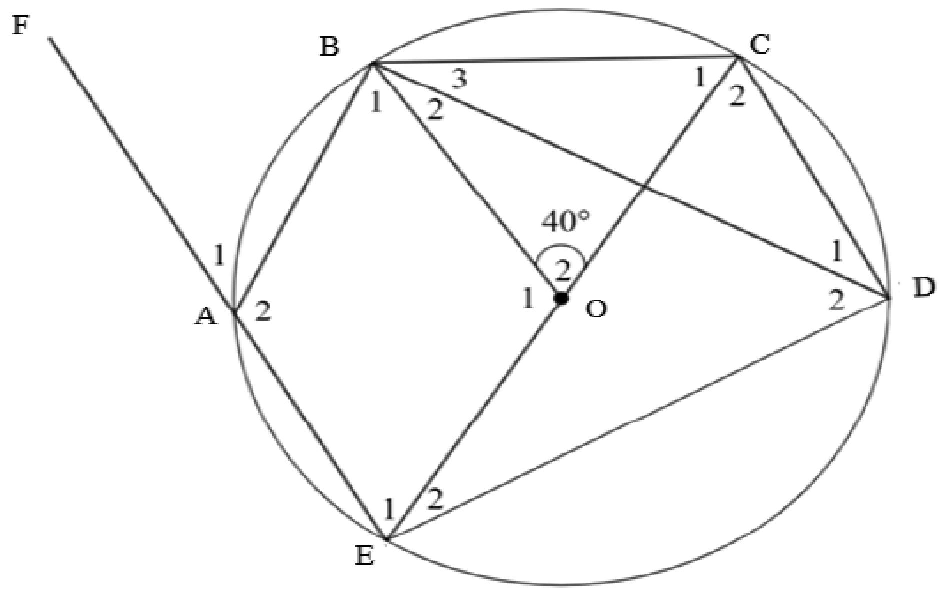
Exercise 11

QUESTION 8

8.1 Complete the following statement:

"An exterior angle of a cyclic quadrilateral is ... to the interior opposite angle". (1)

8.2 In the diagram below O is the centre of the circle. A, B, C, D and E are points on the circle circumference. EA is produced to F and $\hat{O}_2 = 40°$.



Determine, with reasons, the sizes of the following angles:

8.2.1 $\hat{D}_2$ (3)

8.2.2 $\hat{A}_1$ (2)

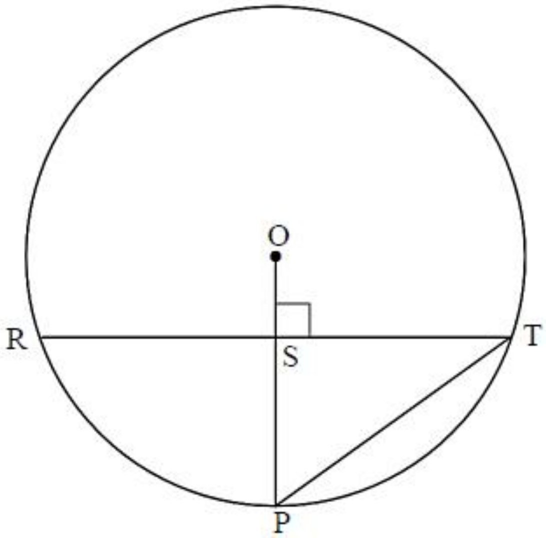
8.2.3 $\hat{D}_1$ (3)

[9]

Exercise 12

QUESTION 6

In the diagram below, O is the centre of the circle with radius OP.
RT = 24 cm, OS = 5 cm and OP $\perp$ RT.



6.1 Complete the following statement:

The line drawn from the (i) ... of a circle perpendicular to a chord (ii) ... the chord. (2)

6.2 Determine, stating reasons, the length of:

6.2.1 SP (4)

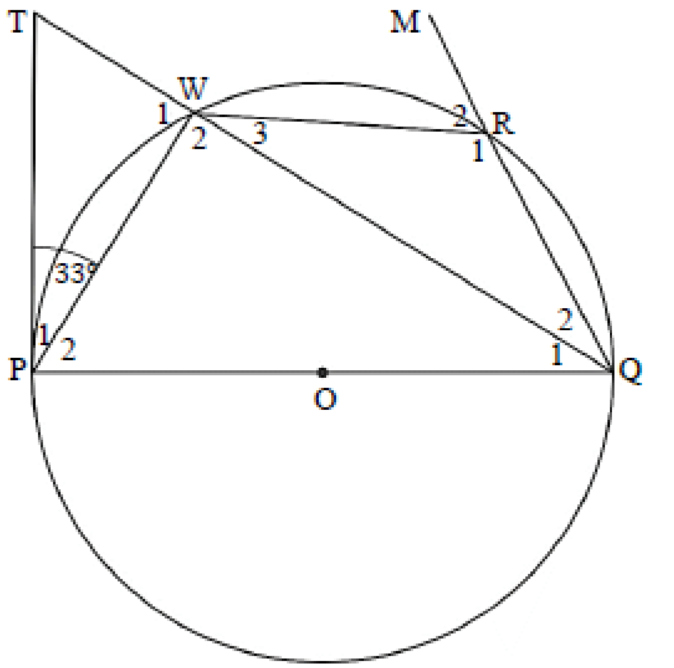
6.2.2 PT (3)

[9]

Exercise 13

QUESTION 7

In the diagram below O is the centre of the circle with P, Q, R and W points on the circumference of the circle. PT is a tangent to the circle at P. PQ is a diameter to the circle and QR extends to M. $\hat{P}_1 = 33°$.



7.1 Complete the following statement:

The (i) ... angle of a cyclic quadrilateral is equal to the (ii) ... opposite angle. (2)

7.2 State the reason why PT is a tangent to the circle. (1)

7.3 Calculate, stating reasons, the sizes of the following angles:

7.3.1 $\hat{Q}_1$ (2)

7.3.2 $\hat{R}_2$ (2)

7.3.3 $\hat{T}$ (2)

7.3.4 $\hat{Q}_2$ if RW = RQ (3)

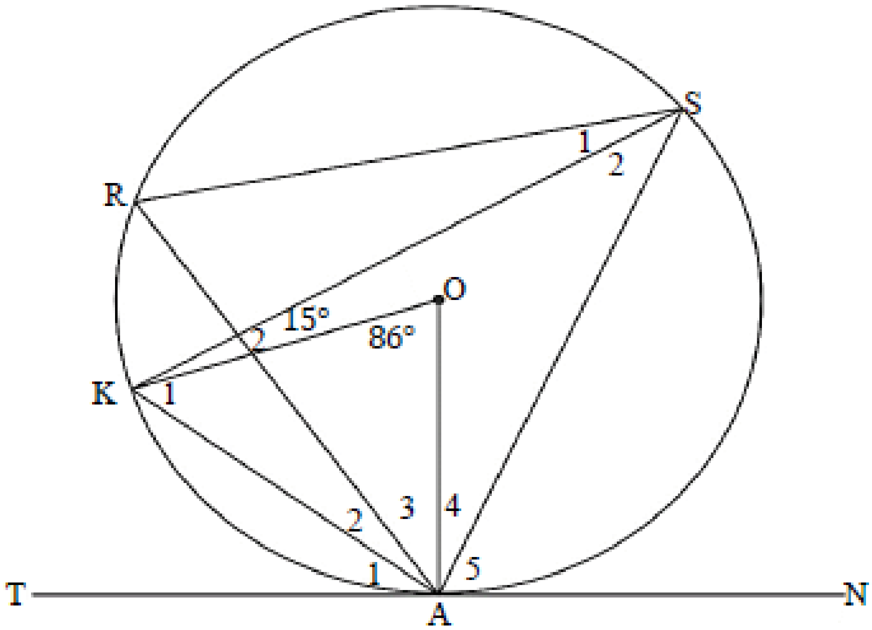
7.4 Show that $\Delta TQP \,|||\, \Delta PQW$ (4)

[16]

Exercise 14

QUESTION 8

In the diagram below O is the centre of the circle AKRS. Tangent TAN touches the circle at point A. $\hat{O} = 86°$ and $\hat{K}_2 = 15°$



Calculate, stating reasons, the sizes of the following angles:

8.1 $\hat{A}_1$ (4)

8.2 $\hat{K}_1$ (2)

8.3 $\hat{R}$ (3)

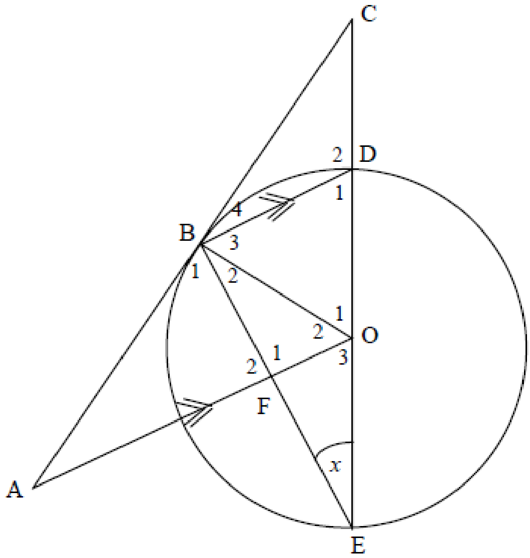
8.4 $\hat{A}_4$ (3)

8.5 Given: M is the midpoint of RA and is the point intersection of RA and KO.
Calculate, stating reasons, the size of $\hat{A}_3$ (3)

[15]

Exercise 15

8.2 ED is a diameter of the circle, with centre O. ED is extended to C. CA is a tangent to the circle at B. AO intersects BE at F. BD || AO. $\hat{E} = x$.



8.2.1 Write down, with reasons, THREE other angles equal to x. (4)

8.2.2 Determine, with reasons, $C\hat{B}E$ in terms of x. (3)

8.2.3 Prove that F is the midpoint of BE. (4)

SECTION 3: Calculus

1. LIMITS AND FUNTIONAL NOTATION

A limit is a value that a function gets closer to when the input gets closer to a certain value. When finding a limit, we look at what happens to a function value of a curve (the y – value) as we get closer and closer to a specific x – value on the curve.

When writing a limit, we use the notation $\lim_{x \to c} f(x)$, where 'lim' indicates that we are determining the limit, the notation $x \to c$ informs us which specific value the x – value is approaching and $f(x)$ represents the function with which we are working.

Substitute c into $f(x)$, taking into consideration that the function does not become undefined. In the case of UNDEFINED, simplify $f(x)$ first then substitute c.

$$\lim_{x \to 0} \frac{2x^2 + 4x}{x}$$

$$\lim_{x \to 0} \frac{2x^2}{x} + \frac{4x}{x}$$

$$\lim_{x \to 0} 2x + 4$$

$$= 2(0) + 4$$
$$= 4$$

Functional notation:

- Remember that $f(x)$ is read as f of x.
- By substituting values of x into the function $f(x)$, one can obtain a set of coordinates $(x\ ;\ y)$ or $(x\ ;\ f(x))$

WORKED EXAMPLE

1. Given $f(x) = 2x + 1$, calculate:
   a) $f(4)$
   b) $f(x + 1)$

Solutions:

a) $f(4) = 2(4) + 1$

$= 9$

b) $f(x + 1) = 2(x + 1) + 1$

$= 2x + 2 + 1$

$= 2x + 3$

ACTIVITY 1

1. Simplify:

   1.1. $\lim_{x \to 4} 1 - 3x$

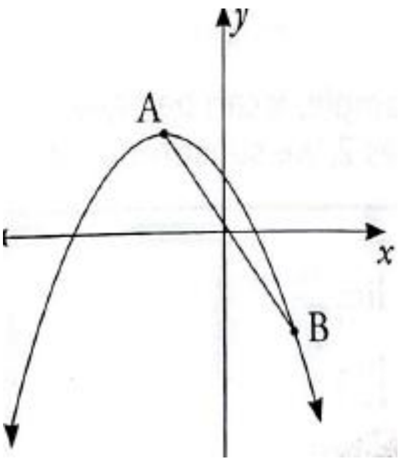
   1.2. $\lim_{x \to 1} \frac{x^2 + 3x - 4}{x - 1}$

2. Given $f(x) = x^2 - 4$, determine $f(x + 1)$.

2. AVERAGE GRADIENT

The gradient of a linear function is simply determined by $m = \frac{y_2 - y_1}{x_2 - x_1}$ but when we work with curves, the gradient changes at each point along the curve. So, we need to determine the average gradient.

Average gradient between two point on a curve is the gradient of the straight line passing through those two points.



Therefore, the average gradient will be: $m_{ave} = \frac{f(b) - f(a)}{b - a}$

WORKED EXAMPLE

Find the average gradient of $f(x) = x^2 - 16$ between $x = 5$ and $x = 9$.

Solution:

$$m_{ave} = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{(9)^2 - 16 - ((5)^2 - 16)}{9 - 5}$$

$$= \frac{65 - 9}{4}$$

$$= 14$$

ACTIVITY 2

1. Determine the average gradient of $g(x) = x^3 - 7x^2 - 10x + 16$ between $x = -2$ and $x = 4$.
2. A ball thrown out the window from the third floor of a building has a height of $h(t) = t^2 - 9t + 20$ meters after t seconds. Calculate the average velocity of the ball between 2 sec and 4 sec.

3. FIRST PRINCIPLES

From the graph above, line AB is a secant. If point B is moved towards point A, then the distance (h) between these two points becomes smaller and smaller thus changing line AB from being a secant to being a tangent.

When the latter happens and the two points now share the same position, we no longer talk about the average gradient. We now talk about the instantaneous gradient (gradient at a point), which is also the gradient of the tangent or the DERIVATIVE of a function (the limit of the average gradient).

The derivative can be written as $f'(x)$ and is defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

The process of finding the derivative is called differentiation. When we use the definition above to differentiate a function, it is said that we differentiating using FIRST PRINCIPLES.

- Copy the definition (formula) **EXACTLY AS IT IS** from the formula sheet.
- Substitute $f(x)$ and $f(x + h)$. REMEMBER WHAT YOU LEARNT UNDER FUNCTIONAL NOTATION.
- Simplify by breaking the brackets.
- Simplify by grouping like terms
- Simplify the fraction and write the final answer

WORKED EXAMPLE

Determine the derivative of $f(x) = 1 - x$ using FIRST PRINCIPLES.

Solution:

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{1 - (x+h) - (1-x)}{h} \\
&= \lim_{h \to 0} \frac{1 - x - h - 1 + x}{h} \\
&= \lim_{h \to 0} \frac{-h}{h} \\
&= \lim_{h \to 0} -1 \\
&= -1
\end{aligned}
$$

ACTIVITY 3

1. Determine using FIRST PRINCIPLES the derivative of the following

   1.1. $f(x) = 2 - 3x$

   1.2. $f(x) = 7x + 1$

3. POWER RULE

We will now determine the derivative using the power rule. The following notations are used for the derivative:

$\frac{dy}{dx}$ if $y = \ldots$ is given

$f'(x)$ if $f(x)$ is given

$D_x[\ldots]$

All the above notations say we need to differentiate with respect to x.

Necessary prior knowledge:

- How to multiply brackets
- How change surd form to exponential form: $\sqrt[n]{x^{m}} = x^{\frac{m}{n}}$
- How to remove a variable from being a denominator: $\frac{1}{a^{n}} = a^{-n}$

NOTE: you may not differentiate using the power rule while your expression has brackets, surds and/or your variable is a denominator.

If $f(x) = ax^{n}$, then the derivative will be $f'(x) = anx^{n-1}$ (Power rule)

Simplify the given expression by removing brackets, changing surds and making sure that your variable is not a denominator, then:

Multiply the coefficient by the exponent, then subtract 1 from the exponent

WORKED EXAMPLE

Determine the following

1. $D_x\left[\frac{2(\sqrt{x}+1)}{x}\right]$
2. $\frac{dy}{dx}$ if $y = (x+2)(x+3)$
3. $\frac{dy}{dx}$ if $y = \frac{x^3 - 4x^2 + 3}{x}$
4. $f'(x)$ if $f(x) = \frac{2}{3x} + \frac{x^2}{2}$
3. $\frac{dy}{dx}$ if $ys = \sqrt{x}$ -5

Solution:

1. $D_x\left[\frac{2\left(x^{\frac{1}{2}}+1\right)}{x}\right]$

$$D_x\left[\frac{2x^{\frac{1}{2}} + 2}{x}\right]$$

$$D_x\left[2x^{\frac{1}{2}} + 2x^{-1}\right]$$

$$= x^{-\frac{1}{2}} - 2x^{-1}$$

2. $\dfrac{dy}{dx}$ if $y = (x+2)(x+3)$

$y = x^2 + 5x + 6$

$\dfrac{dy}{dx} = 2x + 5$

3. $y = \dfrac{x^3}{x} - \dfrac{4x^2}{x} + \dfrac{3}{x}$

$= x^2 - 4x + 3x^{-1}$

$\dfrac{dy}{dx} = 2x - 4 - 3x^{-2}$

4. $f(x) = \dfrac{2}{3}(x^{-1}) + \dfrac{x^2}{2}$

$\dfrac{dy}{dx} = -\dfrac{2}{3}(x^{-2}) + x$

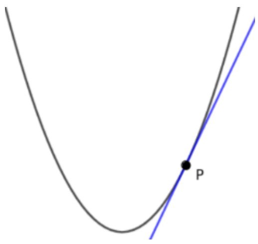
5 $y = \dfrac{\sqrt{x}}{s} - \dfrac{5}{s}$

$\dfrac{dy}{dx} = \dfrac{x^{\frac{1}{2}}}{s}$

ACTIVITY 4

1. Determine $D_y\,[(y + 2)^2]$
2. Given $xy = \sqrt{x^3} - 2$:
   - 2.1. Simplify $xy = \sqrt{x^3} - 2$
   - 2.2. Hence, determine $\frac{dy}{dx}$

4. EQUATION OF THE TANGENT



- P is a point of tangency (point of contact).
- A tangent is a straight line; therefore, we determine its equation easily by using:
  $y - y_1 = m(x - x_1)$
- m is the instantaneous gradient (derivative)
- $(x_1\ ;\ y_1)$ point of tangency

How to determine the equation of a tangent.

Given the function $f(x)$ and the x – coordinate of the point of tangency. Do the following 3 substitutions:

1. Substitute the given x value into the derivative $f'(x)$ to find the gradient m.
2. Substitute the given x value into $f(x)$ to obtain the y – coordinate of the point of tangency $(x_1 ; y_1)$.
3. Substitute the obtained m and $(x_1 ; y_1)$ into $y - y_1 = m(x - x_1)$ and simplify.

WORKED EXAMPLE

Determine the equation of the tangent to the curve $g(x) = 5x^3 + x^2$ at $x = 1$.

Solution:

$g'(x) = 15x^2 + 2x$

$g'(1) = 15(1)^2 + 2(1)$

$= 17$

$g(1) = 5(1)^3 + (1)^2$

$= 6 \qquad (1 ; 6)$

$y - 6 = 17(x - 1)$

$y = 17x - 17 + 6$

$y = 17x - 11$

How to determine the coordinates of the point of tangency (point of contact).

Given the function $f(x)$ and the gradient. Do the following 2 substitutions:

1. Substitute the given gradient m into the derivative $f'(x)$ to find the x coordinate.
2. Substitute the obtained x coordinate into $f(x)$ to obtain the y – coordinate of the point of tangency $(x_1 ; y_1)$.

WORKED EXAMPLE

Determine the point of contact between the tangent with the gradient of 10 and the curve $f(x) = x^2 + 2x - 1$.

Solution:

$f\,'(x) = 2x + 2$

$10 = 2x + 2$

$8 = 2x$

$4 = x$

$f(4) = (4)^2 + 2(4) - 1$

$= 23$

$\therefore$ The point of contact is $(4\ ;\ 23)$

ACTIVITY 5

1. Determine the equation of the tangent to curve of $f(x) = \frac{1}{x} + 1$ at $x = 1$.
2. The tangent with the gradient of 2 touches the curve of $g(x) = x^2 - 8x + 16$. Determine the coordinates of the point of tangency.

5. CUBIC FUNCTION

Sketching the cubic function:

To sketch the cubic function, determine these key features:

- Intercepts (both x and y intercepts)
- Turning points (stationary points)

Sketch the graph, clearly indicating the intercepts with the axis as well as the turning points.

- y intercept

  Let $x = 0$ and simplify.

- $x$intercept

  Let $y = 0$ and solve a third-degree polynomial by inspection, long division or synthetic method. The latter will require you to use FACTOR THEOREM to find the divisor (factor).

  NOTE: there can be one, two or three x intercepts depending on the given function.

- Stationary / turning points of a function $f(x)$
  - Determine the derivative $f\,'(x)$
  - Equate the derivative to zero, $f\,'(x) = 0$
  - Solve for x, then substitute the obtained values of x into the original function $f(x)$ to obtain the corresponding y values.
- Plot the points and join them

WORKED EXAMPLE

Given the function $f(x) = -x^3 + 5x^2 + 8x - 12$:

1. Determine the y intercept of f.
2. Show that $x - 1$ is a factor of f.
3. Hence, determine the $x - int$ of f.
4. Determine the coordinates of the turning points.
5. Sketch the graph of f, clearly indicate the intercepts with the axis as well as the turning points.

Solution:

1. $f(0) = -(0)^3 + 5(0)^2 + 8(0) - 12$

   $= -12$

2. $x - 1 = 0$

   $x = 1$

   $f(1) = -(1)^3 + 5(1)^2 + 8(1) - 12$

   $= 0$

   $\therefore x - 1$ is a factor.

3. $$1 \begin{array}{|rrrr|} -1 & 5 & 8 & -12 \\ 0 & -1 & 4 & 12 \end{array}$$

   $(x - 1)(-x^2 + 4x + 12) = 0$

   $x = 1$ or $x = \frac{-4 \pm \sqrt{(4)^2 - 4(-1)(12)}}{2(-1)}$

   $x = 1 \quad or \quad x = -2 \quad or \quad x = 6$

   $(1\ ;\ 0) \quad (-2\ ;\ 0) \quad (6\ ;\ 0)$

4. $f'(x) = -3x^2 + 10x + 8$

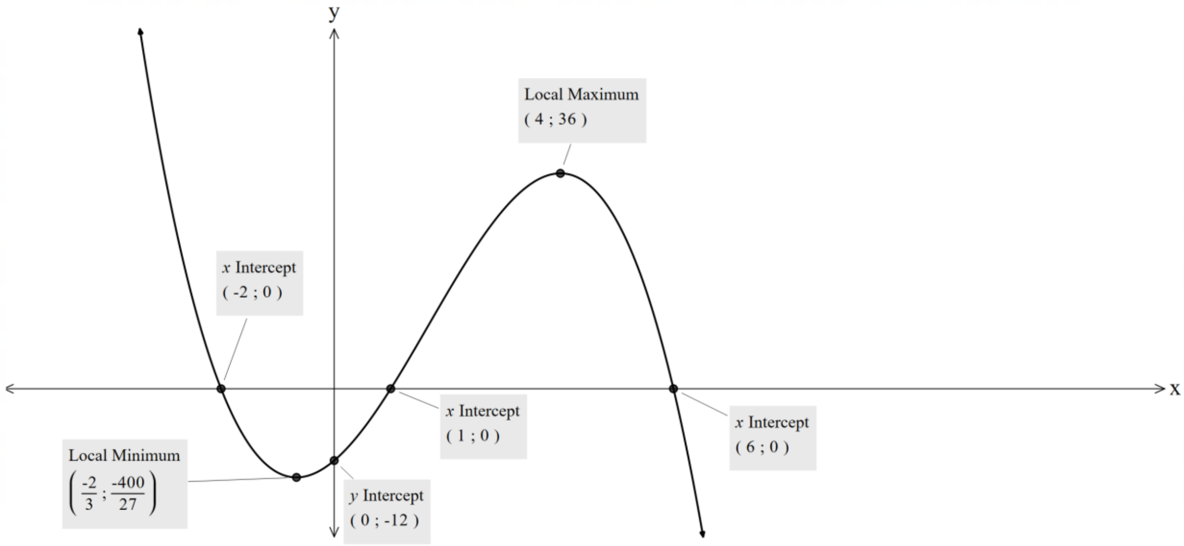
   $-3x^2 + 10x + 8 = 0$

   $$x = \frac{-10 \pm \sqrt{(10)^2 - 4(-3)(8)}}{2(-3)}$$

   $x = -\frac{2}{3} \quad or \quad x = 4$

   $\left(-\frac{2}{3}\ ;\ -\frac{400}{27}\right) \qquad (4\ ;\ 36)$

5.



ACTIVITY 6

Given the function $f(x) = x^3 - 6x^2 + 9x$:

1. Determine the y intercept of f.
2. Show that $x - 3$ is a factor of f.
3. Hence, determine the $x - int$ of f.
4. Determine the coordinates of the turning points.
5. Sketch the graph of f, clearly indicate the intercepts with the axis as well as the turning points.

6. APPLICATION OF CALCULUS

Calculus can be used to solve optimisation and rate of change problems, this means we will have to find an optimum. The optimum value can be either a minimum or a maximum. The same principle used to determine stationary points will be used to determine the optimum value.

WORKED EXAMPLE

The displacement of a ball in meters from the centre of the playground after t seconds is given by $s(t) = 2t^2 - 25t + 150$

1. What is the initial position of the ball from the centre?
2. How fast was the ball moving at exactly 6 sec?

3. What is the minimum distance between the ball and the centre of the playground?

Solution:

1. $s(0) = 2(0)^2 - 25(0) + 150$

   $= 150$

2. $s'(t) = 4t - 25$

   $s'(6) = 4(6) - 25$

   $= -1\ m/s$

3. $0 = 4t - 25$

   $4t = 25$

   $t = \dfrac{25}{4}$

$$s\left(\frac{25}{4}\right) = 2\left(\frac{25}{4}\right)^2 - 25\left(\frac{25}{4}\right) + 150$$

$$= 71{,}88\ m$$

ACTIVITY 7

A tourist travels in a car over a mountainous pass during his trip. The height of the car above sea level, after t minutes, is given as $s(t) = 5t^3 - 65t^2 + 200t + 100$ meters. The journey lasts 8 minutes.

1. How high is the car above sea level when it starts its journey on the mountainous pass?
2. Calculate the car's average rate of change of height above sea level with respect to time in the first 2 minutes of the journey in the mountainous pass.
3. Calculate the velocity of the car 4 minutes after starting the journey on the mountainous pass.
4. After how many minutes will car reach its maximum height above sea level?
5. Hence, determine the maximum height that the car will reach.
6. How many minutes after the journey has started will the rate of change of height with respect to time be a minimum?

EXERCISES

Average gradient

Exercise 1

2.8 Given $f(x) = -x^2 + 9$. The points $A(-2;5)$, $B(-1;8)$ and $C(2;5)$ are on the graph of $f(x)$.

2.8.1 Calculate the average gradient

2.8.1.1 from A to B

2.8.1.2 from B to C and

2.8.1.3 from A to C.

2.8.2 Which of the average gradients is the smallest?

2.9 Given $f(x) = x^3 + 7x^2 + 7x - 15$. Calculate the average gradient between the points where $x = -1$ and $x = 2$.

2.10 Calculate the average gradient for the function given by $f(x) = x^3 - 1$ between the points $(1;0)$ and $(2;7)$.

Differentiation : Rules and first principle

Exercise 1

QUESTION 6

6.1 Determine $f^{/}(x)$ using FIRST PRINCIPLES if $f(x) = 7x - 2$ (5)

6.2 Determine:

6.2.1 $\frac{d}{dx}\left(\pi^2\right)$ (1)

6.2.2 $D_x\left(x^4 - \sqrt[3]{x}\right)$ (3)

6.2.3 $\frac{dy}{dx}$ if $y = \frac{x^5 + 2}{x^2}$ (4)

Exercise 2

QUESTION 6

6.1 Determine $f^{/}(x)$ using FIRST PRINCIPLES if $f(x) = \frac{1}{2}x$ (4)

6.2 Determine EACH of the following:

6.2.1 $\frac{dA}{dr}$ if $A = \pi r^2$ (1)

6.2.2 $D_x\left[\left(x - \sqrt{x}\right)^2\right]$ (5)

Exercise 3

QUESTION 6

6.1 Determine $f^{/}(x)$ using FIRST PRINCIPLES if $f(x) = 2x + 3$ (5)

6.2 Determine:

6.2.1 $\frac{dy}{dx}$ $if\ y = -x^{-5} + 3x^{4}$ (2)

6.2.2 $f^{/}(x)$ if $f(x) = \frac{3}{x^4} - \frac{x}{\sqrt{x}}$ (4)

6.2.3 $D_x\left[\frac{x^2 + x - 6}{x + 3}\right]$ (3)

Exercise 4

QUESTION 6

6.1 Determine $f^{/}(x)$ using FIRST PRINCIPLES if $f(x) = 5 + x$ (5)

6.2 Determine:

6.2.1 $\frac{dy}{dx}$ if $y = x(x + 9)$ (3)

6.2.2 $D_x[\sqrt[7]{x} + \pi p^3]$ (3)

6.2.3 $f^{/}(x)$ if $f(x) = \frac{1 - x^9}{x^2}$ (4)

Equation of the tangent

Exercise 1

6.3 The tangent to the curve of the function defined by $p(x) = x^3 + 1$ passes through point $A(2; k)$.

6.3.1 Calculate the numerical value of k. (2)

6.3.2 Determine $p^{/}(x)$ (1)

6.3.3 Hence, determine the equation of the tangent to the curve of the function at point A. (3)

[19]

Exercise 2

6.3 The equation of a tangent to the curve of function g defined by $g(x) = ax^2 - x$ is $3x - y + 2 = 0$

The point of contact of the tangent and the curve of g is $(-1;-1)$. Determine the numerical value of a. (5)

Exercise 3

6.3 Determine the average gradient of the function defined by $h(x) = -2x^2 + 2$ between the points where $x = 0$ and $x = 2$ (3)

6.4 Given: $g(x) = 1 - x^2$

6.4.1 Determine the gradient of the tangent to g at the point where $x = -3$ (2)

6.4.2 Hence, determine the equation of a tangent to g at the point where $x = -3$ (3)

Exercise 4

QUESTION 6

6.3 Given $g(x) = -4x^2$

Determine:

6.3.1 $g(2)$ (1)

6.3.2 The equation of the tangent to g at a point where $x = 2$ (4)

Cubic function

Exercise 1

QUESTION 7

Given: $f(x) = -x(x-3)(x-3)$

7.1 Write down the coordinates of the x-intercepts of f. (2)

7.2 Write down the y-intercept of f. (1)

7.3 Show that $f(x) = -x^3 + 6x^2 - 9x$ (2)

7.4 Determine the coordinates of the turning points of f. (5)

7.5 Sketch the graph of f on the ANSWER SHEET provided. Clearly show ALL the intercepts with the axes and the turning points. (4)

7.6 Determine the values of x for which the graph of f is increasing. (2)

[16]

Exercise 2

QUESTION 7

Given: Function f defined by $f(x) = -(x-1)^2(x+3) = -x^3 - x^2 + 5x - 3$

7.1 Write down the y-intercept of f. (1)

7.2 Determine the x-intercepts of f. (2)

7.3 Determine the coordinates of the turning points of f. (5)

7.4 Sketch the graph of f on the ANSWER SHEET provided. Clearly show ALL the intercepts with the axes and the turning points. (4)

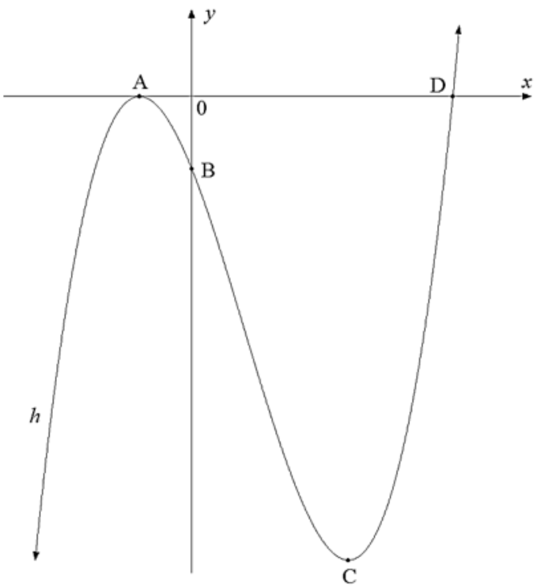
7.5 Determine the values of x for which $f'(x) > 0$ (2)

[14]

Exercise 3

The graph below represents the function defined by $h(x) = x^3 - 3x^2 - 9x - 5$

7.1
- A and C are the turning points of h.
- A, B and D are intercepts on the axes.



7.1.1 Write down the coordinates of B. (1)

7.1.2 Show that $x+1$ is a factor of h. (2)

7.1.3 Hence, determine the coordinates of D. (3)

7.1.4 Determine the coordinates of C. (5)

7.2 Write down the values of x for which h is increasing. (2)

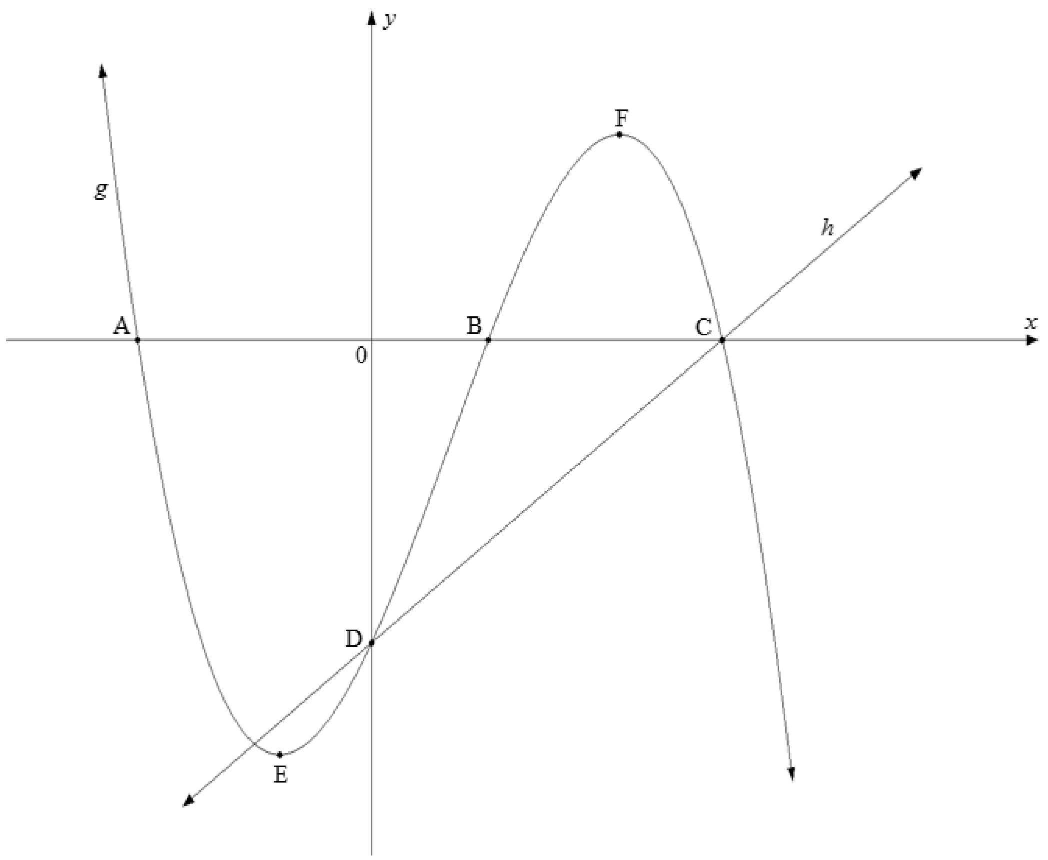
Exercise 4

QUESTION 7

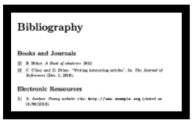
The graphs below represent the functions defined by $g(x) = -(x+2)(x-1)(x-3)$ and $h(x) = 2x + p$

E and F are the turning points of g.

A, B, C and D are intercepts on the axes.



7.1	Write down the coordinates of C.	(2)
7.2	Write down the value of p.	(1)
7.3	Determine the length of AC.	(2)
7.4	Express $g(x) = -(x+2)(x-1)(x-3)$ in the form $g(x) = a \rightleftarrows x^3 + bx^2 + cx + d$	(2)
7.5	Determine the coordinates of E and F.	(5)
7.6	Write down the values of x for which $g(x) > 0$	(3)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