JENN

Training and Consultancy

The path to enlightened education

SUBJECT: MATHEMATICS

GRADE 12

2026 WINTER CLASSES

TEACHER AND LEARNER CONTENT MANUAL

Topic

Differential Calculus and Euclidean Geometry

ICON DESCRIPTION

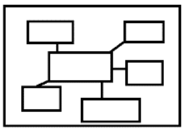
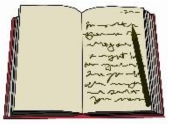
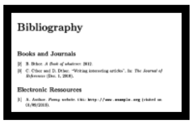
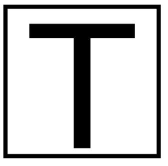
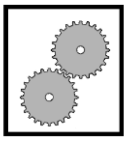
 MIND MAP	 EXAMINATION GUIDELINE	 CONTENTS	 ACTIVITIES
 BIBLIOGRAPHY	 TERMINOLOGY	 WORKED EXAMPLES	 STEPS

Table of Contents

DIFFERENTIAL CALCULUS

The concept of a limit

The limit of a function is the y value to which the graph approaches as the values of x approach a certain value from both the left and the right. Therefore, the limit of a polynomial functions, $f(x)$ as x approaches p is $f(p)$.

The mathematical notation can be expressed as illustrated in the following examples.

Example 1

Determine the limit of $2x^2 - x + 5$ as x approaches 3.

$\lim\limits_{x \to 3}(2x^2 - x + 5)$

$= 2(3)^2 - 3 + 5$

$= 20$

Example 2

Determine the following limit (if it exist): $\lim\limits_{x \to 3}(3x^2 - 2x + 4)$

$\lim\limits_{x \to 3}(3x^2 - 2x + 4)$

$= 3(3)^2 - 2(3) + 4$

$= 25$

Example 3

Simplify: $\lim\limits_{a \to 2} \frac{a^2 - 4}{a - 2}$

In this example, we can see that if we substitute 2 in the given expression; our answer is undefined as division by 0 is undefined. This therefore tell us that we need to do some process(es) (factorisation of the numerator in this case) before we go on to substitution.

$\lim\limits_{a \to 2} \frac{a^2 - 4}{a - 2}$

$= \lim\limits_{a \to 2} \frac{(a-2)(a+2)}{a - 2}$

$= \lim\limits_{a \to 2}(a + 2)$

$= 2 + 2$

$= 4$

***Key take away**: From the above 3 examples it is very important to note that the limit can only be dropped when we start substituting the value being approached by the variable in question.*

Exercise 1

Determine the following limits (if they exist)

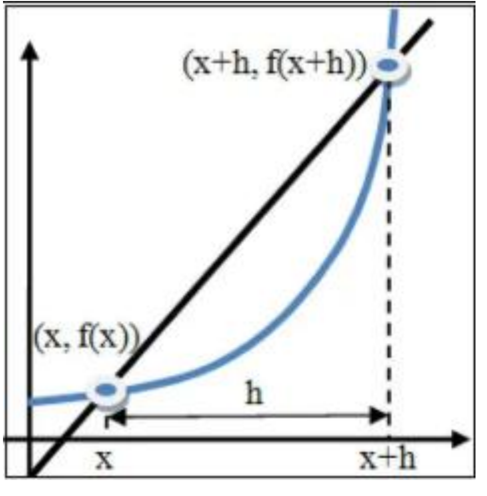
1. $\lim_{x \to 3}(x+5)$
2. $\lim_{p \to 0}(3-2p)$
3. $\lim_{x \to -1} \frac{x^2-1}{x+1}$
4. $\lim_{h \to 0} \frac{h^2}{h}$
5. $\lim_{x \to 1} \frac{x^2+x-2}{x-1}$
6. $\lim_{r \to 1} 5$
7. $\lim_{x \to \frac{1}{2}}(-x^2+3)$
8. $\lim_{x \to -2} \frac{x^3+8}{x+2}$

Average Gradient (Rate of Change) between two Points on a Curve

The average gradient between any two points; $(x_1; y_1)$ and $(x_2; y_2)$ on a curve is the gradient of the straight line passing through the two points. The average gradient can be calculated using the formula:

$$\frac{change\ in\ y}{change\ in\ x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Suppose $(x; f(x))$ and $((x+h); f(x+h))$ are any two points on the curve $y = f(x)$ as shown in the diagram below:



Average gradient

$= \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{f(x+h) - f(x)}{x+h-x}$

$= \frac{f(x+h) - f(x)}{h}$

Key Highlight:

1. $h = x_2 - x_1$
2. As $\left(x; f(x)\right)$ and $\left((x+h); f(x+h)\right)$ - the two points move closer and closer to each other, the value of h gets closer and closer to zero, i.e $h \to 0$
3. The closer h gets to zero, the closer we are to the gradient of a curve at a point or the gradient of a tangent.
4. We use $\lim\limits_{h \to 0}$ to illustrate that the distance between the two points is very, very small.
5. When we use $\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$ to get the gradient; we say we are differentiating by using the definition or differentiating from first principles.

Differentiating from the First Principles (from the Definition)

To differentiate from the first principles, we recommend you follow the following steps:

Step 1: Copy the given function $f(x)$

Step 2: Simplify $f(x+h)$

Step 3: Simplify $f(x+h) - f(x)$

**** Always make sure that $f(x)$ is in brackets*

Step 4: Copy the **CORRECT** formula from the formula sheet; $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Step 5: Substitute answer from ***step 3*** in ***step 4***

Step 6: Simplify ***step 5***

Step 7: **Write your ANSWER with CONFIDENCE**.

Example 1

Find, from first principles, the gradient of $f(x) = 2x$ at any point.

$f(x) = 2x$

$f(x+h) = 2(x+h)$

$\qquad\qquad = 2x + 2h$

$f(x+h) - f(x) = 2x + 2h - (2x)$

$\qquad\qquad = 2h$

$f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim\limits_{h \to 0} \frac{2h}{h}$

$f'(x) = \lim\limits_{h \to 0} 2$

$f'(x) = 2$

Example 2

Find, from first principles, the derivative of $f(x) = 6$ at any point.

$f(x) = 6$

$f(x+h) = 6$ (*since there is no x in the given function it therefore means $f(x+h) = f(x)$*)

$f(x+h) - f(x) = 6 - 6$

$= 0$

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h \to 0} \frac{0}{h}$

$f'(x) = \lim_{h \to 0} 0$

$f'(x) = 0$

Example 3

Find $f'(x)$ from first principles if $f(x) = 3x^2 + x$

$f(x) = 3x^2 + x$

$f(x+h) = 3(x+h)^2 + (x+h)$

$= 3(x^2 + 2xh + h^2) + x + h$

$= 3x^2 + 6xh + 3h^2 + x + h$

$f(x+h) - f(x) = 3x^2 + 6xh + 3h^2 + x + h - (3x^2 + x)$

$= 3x^2 + 6xh + 3h^2 + x + h - 3x^2 - x$

$= 6xh + 3h^2 + h$

$= h(6x + 3h + 1)$

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h \to 0} \frac{h(6x+3h+1)}{h}$

$f'(x) = \lim_{h \to 0} (6x + 3h + 1)$

$f'(x) = 6x + 3(0) + 1$

$f'(x) = 6x + 1$

Example 4

Determine, from the first principles, the derivative of $f(x) = \frac{1}{x}$.

$f(x) = \frac{1}{x}$

$f(x+h) = \frac{1}{x+h}$

$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x}$

$= \frac{1(x)-1(x+h)}{(x+h)x}$

$= \frac{x-x-h}{x^2+xh}$

$= \frac{-h}{x^2+xh}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{\left(\frac{-h}{x^2+xh}\right)}{h}$

$f'(x) = \lim_{h\to 0} \left(\frac{-h}{x^2+xh} \times \frac{1}{h}\right)$

$f'(x) = \lim_{h\to 0} \frac{-1}{x^2+xh}$

$f'(x) = \frac{-1}{x^2+x(0)}$

$f'(x) = \frac{-1}{x^2}$

Exercise 2

1. $f(x) = 5$
2. $f(x) = mx + c$
3. $f(x) = -x^2 - x$
4. $f(x) = -\frac{1}{2}x^2 - 3x$
5. $f(x) = px^2 - qx + r$
6. $f(x) = \frac{1}{2}x^2 - 3x + 6$
7. $f(x) = 2x^3$
8. $f(x) = -\frac{2}{x}$

****Concentration to be put on constant functions $f(x) = c$, linear functions $f(x) = bx + c$, quadratic functions $f(x) = ax^2 + bx + c$, cubic functions $f(x) = ax^3$ and the rational functions $f(x) = \frac{a}{x}$. The constants a, b and c can be any constant variable or whole numbers or fractions or any type of numbers.*

Rules of Differentiation

***Rule 1*: If $\boldsymbol{f(x) = ax^n}$, then $\boldsymbol{f'(x) = a.nx^{n-1}}$**

Example

$f(x) = -4x^3$, then $\boldsymbol{f'(x)} = -12x^2$

***Rule 2*: If $\boldsymbol{f(x) = ax}$, then $\boldsymbol{f'(x) = a}$**

Example

$f(x) = -4x$, then $\boldsymbol{f'(x)} = -4$

***Rule 3*: If $\boldsymbol{f(x)} =$ number, then $\boldsymbol{f'(x) = 0}$**

Example

$f(x) = 9$, then $\boldsymbol{f'(x)} = 0$

***Rule 4*:** If $f(x) =$ letter (not x) which is a treated as a constant, then $\boldsymbol{f'(x) = 0}$

Example

If $f(x) = k$, then $f^{/}(x) = 0$

***Rule 5*:** $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$

Example

Determine $\frac{dy}{dx}$ if $y = -4x^2 + 7x - 8$, then $\frac{dy}{dx} = -8x + 7$

Exponents, Surds and Definitions Relevant for Differentiation using Rules

Before discussing the examples that follow, it is critical that you go over the exponent and surd laws from grade 10 and 11. The following definitions will also prove helpful to you as we continue our journey through this awesome module.

1. $\frac{x^n}{a} = \frac{1}{a}x^n$
2. $\frac{a}{x^n} = ax^{-n}$
3. $\frac{a}{bx^n} = \frac{a}{b}x^{-n}$
4. $\frac{a}{(bx)^n} = a\ .\ (bx)^{-n}$
5. $\sqrt[n]{ax^m} = \sqrt[n]{a}\ .\ x^{\frac{m}{n}}$
6. $\frac{a}{\sqrt[n]{x^m}} = a\ .\ x^{-\frac{m}{n}}$

There are different symbols which are used to represent gradient. We will now discuss some basic examples to illustrate the rules of differentiation as well the different symbols used.

Example 1 (Using the symbols $f'(x)$ and $\frac{dy}{dx}$)

a) If $f(x) = x^4 + 6x + 7$, determine $f'(x)$, the derivative of f.

Rule 1 **Rule 2** **Rule 3**

$$f'(x) = 4x^{4-1} + 6 + 0$$

$$\therefore f'(x) = 4x^3 + 6$$

b) If $g(x) = 4x^5 + 5\pi + t^2$, determine $g'(x)$, the gradient of g.

Rule 1 **Rule 3** **Rule 4**

$$g'(x) = 4 \times 5x^{5-1} + 0 + 0$$

$$\therefore g'(x) = 20x^4$$

c) If $y = \frac{1}{3}x^{-6} + x + \frac{x^3}{3}$, determine $\frac{dy}{dx}$, the derivative of y with respect to x.

$y = \frac{1}{3}x^{-6} + x + \frac{x^3}{3}$

Rewrite $\frac{x^3}{3}$ as $\frac{1}{3}x^3$ clearly separating the coefficient and our respected variable

$\therefore y = \frac{1}{3}x^{-6} + x + \frac{1}{3}x^3$

Now use the rules to differentiate (find the derivative)

$\frac{dy}{dx} = \frac{1}{3} \times -6x^{-6-1} + 1 + \frac{1}{3} \times 3x^{3-1}$ (Rule 1, Rule 2, Rule 1)

$\therefore \frac{dy}{dx} = -2x^{-7} + 1 + x^2$

Note: The symbols $f'(x)$ and $\frac{dy}{dx}$ refer to the gradient or derivative of functions written in the form $f(x) = \ldots$ and $y = \ldots$

Example 2 (Using the symbol D_x)

Determine $D_x\left[\pi x^2 + \pi^2 + x^{-2}\right]$

$D_x\ [\pi x^2 + \pi^2 + x^{-2}]$

$= \pi \times 2x^{2-1} + 0 + (-2)x^{-2-1}$

$= 2\pi x - 2x^{-3}$

Note: Be aware of the placement of the equal signs and the use of the symbols in the previous examples.

More Advanced Types of Differentiation

Remember the following points when differentiating:

1. Make sure that you first remove any:

   - Multiplication of polynomials
   - Fractions (with respected variable on the denominator)
   - Surds

2. Always make sure that all the terms of an expression are in a differentiable form before applying the rules of differentiation to each. This assists to avoid mixing processes.

Example 3

Determine the following:

a) $f'(x)$ if $f(x) = (4x - 3)^2$

Here we first need to expand and simplify so as to remove the brackets.

$f(x) = (4x - 3)^2$

$f(x) = 16x^2 - 24x + 9$

$f'(x) = 16 \times 2x^{2-1} - 24 + 0$

$\therefore f'(x) = 32x - 24$

b) $\frac{dy}{dx}$ if $y = \frac{3}{\sqrt{x}}$

Here we need to first get rid of the surd.

$y = \frac{3}{x^{\frac{1}{2}}}$

We now need to take note of our respected variable which is on the denominator.

$y = 3x^{-\frac{1}{2}}$

$\therefore \quad \frac{dy}{dx} = -\frac{1}{6}x^{-\frac{3}{2}}$

c) $D_x\left[\dfrac{x^2-x-6}{2x^2}\right]$ with 1 term on the denominator; distribute it as shown below.

$= D_x\ \left[\frac{x^2}{2x^2}-\frac{x}{2x^2}-\frac{6}{2x^2}\right]$

$= D_x\ \left[\frac{1}{2}-\frac{1}{2x}-\frac{3}{x^2}\right]$

$= D_x\ \left[\frac{1}{2}-\frac{1}{2}x^{-1}-3x^{-2}\right]$

$= 0-\frac{1}{2}(-1)x^{-1-1}-3(-2)x^{-2-1}$

$= \frac{1}{2}x^{-2}+6x^{-3}$

d) $D_x\left[\dfrac{3x^3-7x^2-6x}{3-x}\right]$ with more than 1 term on the denominator; there is need to factorise the numerator.

$= D_x\ \left[\frac{x(3x^2-7x-6)}{3-x}\right]$

$= D_x\ \left[\frac{x(3x+2)(x-3)}{3-x}\right]$

$= D_x\ \left[\frac{x(3x+2)(x-3)}{-(x-3)}\right]$

$= D_x\ [-x(3x+2)]$

$= D_x\ [-3x^2-2x]$

$= -6x-2$

Exercise 3

Question 1

Determine the derivative, $f'(x)$, in each case:

1.1 $f(x)=x^{13}-15x^2+2x+3$ $\qquad$ $f(x)=1-x+3x^4$

Question 2

2.1 $D_x\left[(2x-3)(x+4)\right]$

2.2 $D_x\left[\dfrac{2}{3x^4}\right]$

2.3 $D_x\left[\sqrt[3]{x}+\dfrac{1}{\sqrt{x}}\right]$

2.4 $D_x\left[8\sqrt{x}-\dfrac{1}{2x}+3m^4\right]$

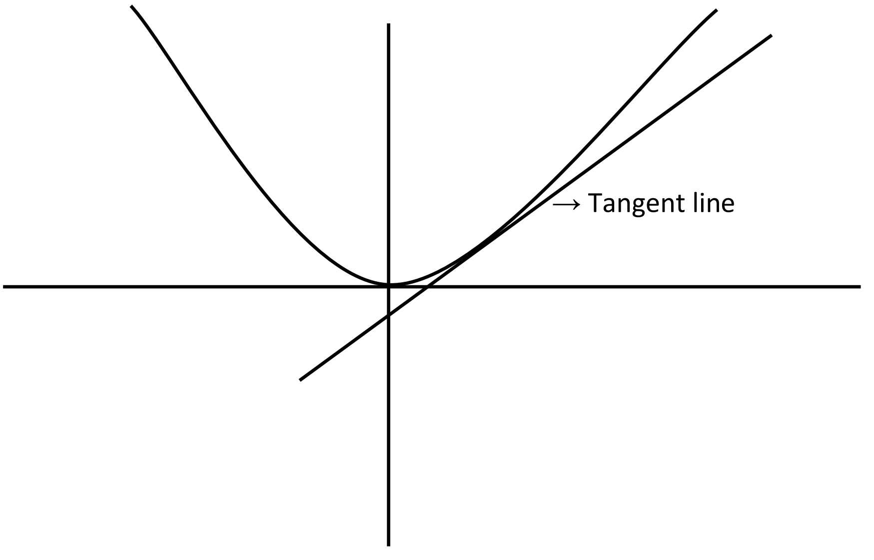
2.5 $D_t\left[\dfrac{t^3-t^2+1}{t}\right]$

2.6 $\dfrac{dy}{dx}$ at $x=1$ if $y=\dfrac{9-x}{x^3}$

2.7 $\dfrac{dy}{dx}$ if $\dfrac{y}{2x}=(1-x)^2$

The Equations of a Tangent to The Curve

A tangent is a straight line that touches a curve at one point. A central principle of calculus is that at any point $x = a$ on the graph of a function f, the gradient of f at $x = a$ is the same as the gradient of the tangent to f at $x = a$. In other words, $m_t = f'(a)$.



Note: The curve and its tangent line have the same gradient at the point of contact.

Given the function $f(x)$ and then required to get the equation of tangent at $x = a$, we recommend following these steps:

Step 1: Substitute $x = a$ in $f(x)$ to get the corresponding y −coordinate; to get point $(a; f(a))$

Step 2: Determine $f'(x)$ using the rules of differentiation

Step 3: Substitute $x = a$ in $f'(x)$ to get the gradient, $m = f'(a)$

Step 4: Determine the **CORRECT EQUATION** of the tangent and be happy you mastered it.

Example 1

Determine the equation of a tangent to $f(x) = x^2 + x$ at the point (1; 2)

$f'(x) = 2x + 1$

$m = f'(1) = 2(1) + 1$

$m = 3$

$y - y_1 = m(x - x_1)$

$y - 2 = 3(x - 1)$

$\therefore \quad y = 3x - 1$

Example 2

The graph of $g(x) = -3x^3 + 2x^2 - 3x + 5$ is given.

Determine the equation of the tangent to the graph of g at the point where $x = -1$.

$g(-1) = -3(-1)^3 + 2(-1)^2 - 3(-1) + 5$

$= 13 \qquad\qquad\qquad (-1; 13)$

$g(x) = -3x^3 + 2x^2 - 3x + 5$

$\therefore g'(x) = -9x^2 + 4x - 3$

$m = g'(-1) = -9(-1)^2 + 4(-1) - 3$

$m = -16$

Therefore, the equation of the tangent is:

$y - y_T = m_t(x - x_T)$

$y - 13 = -16(x - (-1))$

$y - 13 = -16(x + 1)$

$y - 13 = -16x - 16$

$y = -16x - 3$

Exercise 4

1. Find the equation of the tangent to the following curves at a given value of x.

   1.1. $y = x^2 - x - 6$ at $x = 1$

   1.2 $y = -\dfrac{6}{x}$ at $x = -2$

2 The equation of the tangent to $f(x) = 3x^2 + bx + c$ at $x = 1$ is given by $y = 9x - 9$.

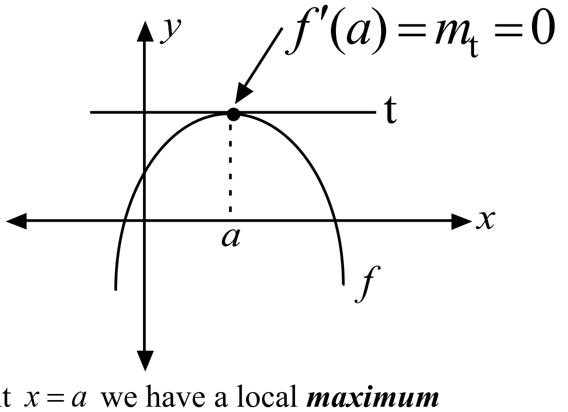
Determine the values of b and c.

3 For which value of x is the tangent to the curve $y = x^2 + 3x - 5$ parallel to the line $3y + 2x = 6$?

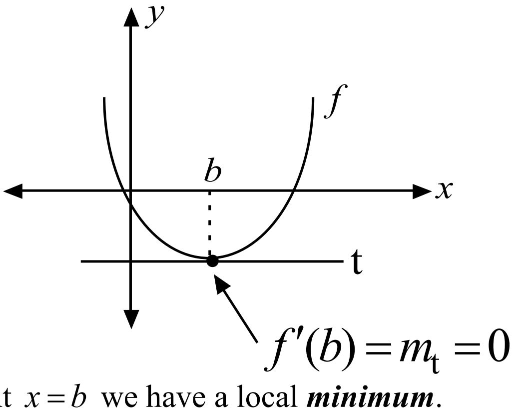
Cubic Functions

Coordinates of Turning Points (Stationary Points)

At any ***stationary point*** on a graph *f*, the gradient is zero. If $x = a$ is a stationary point, then $f'(a) = 0$ since the gradient of the tangent line at $x = a$ is zero (the line is horizontal). At a stationary point, the function neither increases nor decreases.



At $x = a$ we have a local ***maximum***



At $x = b$ we have a local ***minimum***.

Given the function $f(x)$ and then required to get the co-ordinates of the turning point, we recommend the following steps:

Step 1: Determine $f'(x)$

Step 2: Equate $f'(x)$ to 0, $f'^{(x)} = 0$

Step 3: Solve for x

Step 4: Substitute the values of x (obtained in *Step 3*) in the original function, $f(x)$, to get the corresponding y −values.

Step 5: You now mastered it; you have to write the coordinates of the turning points.

Points of Inflection (Concavity)

Point of inflection

Method 1

- Equating the second derivative to zero. $f^{//}(x) = 0$

1. Equate the second derivative to zero and solve for x. You now have x-value of point of inflection.
2. Substitute the x-value into the original function, e.g $f(x)$, to get the corresponding y-value, and thus the point of inflection.

Method 2

- Using coordinates of the turning points. The point of inflection lies half-way between the turning points of a cubic function
- Find the coordinates of the point of inflection by finding the midpoint of the turning points.

Concave up: $f^{//}(x) > 0$

Concave down: $f^{//}(x) < 0$

Example 1

Consider the cubic function $h(x) = 2x^3 + 4x^2 - 1$.

i. Determine the coordinates of the turning points.

ii. Determine the coordinates of the point of inflection

i. $h(x) = 2x^3 + 4x^2 - 1$

$h'(x) = 6x^2 + 8x$

$6x^2 + 8x = 0$

$2x(3x + 4) = 0$

$2x = 0$ or $3x + 4 = 0$

$x = 0$ or $x = -\frac{4}{3}$

$h(0) = 2(0)^3 + 4(0)^2 - 1$ or $h(-\frac{4}{3}) = 2\left(-\frac{4}{3}\right)^3 + 4\left(-\frac{4}{3}\right)^2 - 1$

$h(0) = -1$ or $h(0) = \frac{37}{27}$

Therefore; coordinates of turning points are:

$(0; -1)$ and $\left(-\frac{4}{3}; \frac{37}{27}\right)$

ii. Coordinates of the point of inflection

$= \left(\frac{0+\left(-\frac{4}{3}\right)}{2}; \frac{-1+\frac{37}{27}}{2}\right)$

$= \left(-\frac{2}{3}; \frac{5}{27}\right)$

Factorizing a Cubic Function

Different methods can be employed. Regardless of the method used to get all factors, it is worth noting that the factor theorem plays a significant role to at least get the first factor.

The factors theorem states that if $f(a) = 0$; if follows that $x - a$ is a factor of $f(x)$.

Examination question can either request only the factorisation of cubic function or to determine the coordinates of the x −intercepts.

Example 2

Given: $f(x) = x^3 - 8x^2 + 5x + 14$.

i. Factorise $f(x)$ completely.

ii. What are coordinates of the turning points?

iii. For which values of x is $f(x)$ concave down?

i. $f(x) = x^3 - 8x^2 + 5x + 14$

Find a number such that if it is substituted in $f(x)$ we obtain 0

$f(2) = (2)^3 - 8(2)^2 + 5(2) + 14$

$f(2) = 0$

$\therefore (x-2)$ is a factor

$f(x) = (x-2)(x^2 + bx - 7)$

Hence;

$bx^2 - 2x^2 \equiv -8x^2$

$b - 2 = -8$

$b = -6$

$f(x) = (x-2)(x^2 - 6x - 7)$

$f(x) = (x-2)(x-7)(x+1)$

ii. $f(x) = x^3 - 8x^2 + 5x + 14$

$f'(x) = 3x^2 - 16x + 5$

$3x^2 - 16x + 5 = 0$

$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(3)(5)}}{2(3)}$

$x = \frac{1}{3}$ or $x = 5$

$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - 8\left(\frac{1}{3}\right)^2 + 5\left(\frac{1}{3}\right) + 14$ or $f(5) = (5)^3 - 8(5)^2 + 5(5) + 14$

$f\left(\frac{1}{3}\right) = \frac{400}{27}$ or $f(5) = -36$

Therefore; coordinates of turning points are:

$\left(\frac{1}{3}; \frac{400}{27}\right)$ and $(5; -36)$

iii. $f''(x) < 0$

$6x - 16 < 0$

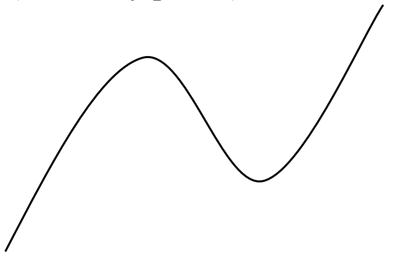
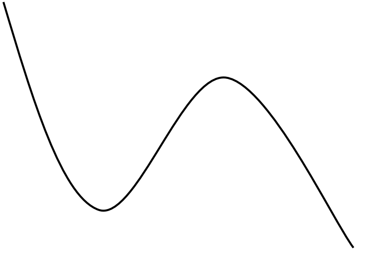
$6x < 16$

$x < \frac{8}{3}$

Sketching the Cubic Function

The standard form of a cubic function is given by: $y = ax^3 + bx^2 + cx + d$

Basic shapes of a cubic function.

<table>
<tr><th>Shape if $a > 0$</th><th>Shape if $a < 0$</th></tr>
<tr><td>Local maximum and minimum turning points (stationary points).
</td><td>Local maximum and minimum turning points (stationary points).
</td></tr>
</table>

Steps to follow when sketching:

N.B sometimes, if not most of the time, you will be directed by the question as to where to start

1) Determine the x - and y - intercepts
   a) For x intercept:
      i) Let y = 0
      ii) Find the factors and solve for x. (Factor theorem is important).
      iii) Calculate the corresponding y values.
   b) For y intercept: Let $x = 0$ and solve for y
2) Determine the turning points
3) Sketch

Example 3

Given: $f(x)=x^3+2x^2-7x+4$

1 Show that $(x-1)$ is a factor of $f(x)$.

2 Hence, or otherwise, find the x-intercepts of f.

3 Determine the coordinates of the turning points of f.

4 Sketch the graph of f on the ANSWER SHEET provided. Clearly show ALL the intercepts with the axes and the turning points.

Solution

1

$$f(x)=x^3+2x^2-7x+4$$
$$f(1)=(1)^3+2(1)^2-7(1)+4$$
$$\therefore f(1)=0$$
$$\therefore x-1 \text{ is a factor of } f$$

2

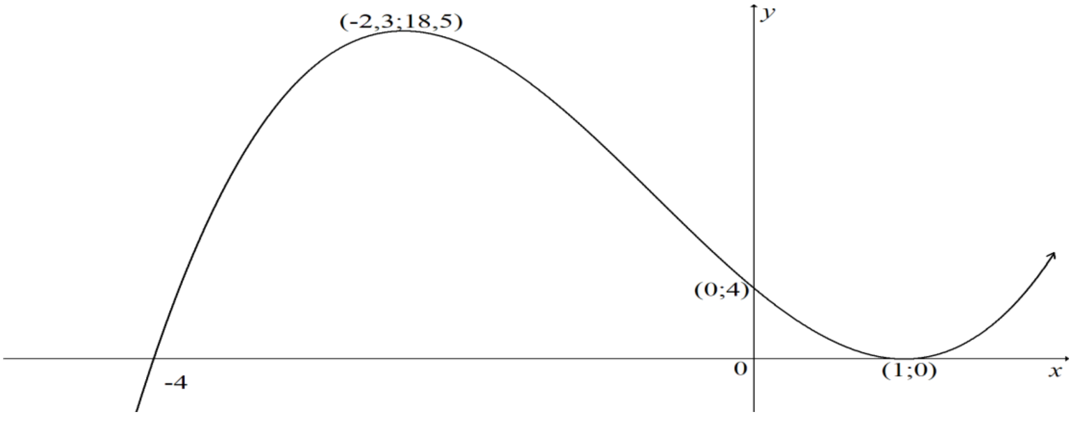
x-intercepts:

$$f(x)=0$$
$$x^3+2x^2-7x+4=0$$
$$(x-1)(x^2+3x-4)=0$$
$$(x-1)(x-1)(x+4)=0$$
$$x=1 \text{ or } x=-4$$

3

$$f(x)=x^3+2x^2-7x+4$$
$$f^{/}(x)=3x^2+4x-7$$
$$0=(3x+7)(x-1)$$
$$\therefore \quad x=-\frac{7}{3} \quad \text{or} \quad x=1$$
$$\left(-\frac{7}{3};\frac{500}{27}\right) \qquad (1;0)$$

4



Exercise 5

QUESTION 1

1. Sketch the graph of $f(x) = x^3 - 12x^2 + 36x$. Show clearly all the intercepts with the axes as well as the coordinates of the stationery points.

QUESTION 2

Given: $f(x) = x^3 + 4x^2 - 7x - 10$

2.1 Write down the y - intercept of f.

2.2 Show that 2 is a root of the equation $f(x) = 0$.

2.3 Hence, factorise $f(x)$ completely.

2.4 If it is further given that the coordinates of the turning points are approximately at $(0{,}7\ ;\ -12{,}6)$ and $(-3{,}4\ ;\ 20{,}8)$, draw a sketch graph of f and label all intercepts and turning points.

2.5 Use your graph to determine the values of x for which $f^{/}(x) < 0$

QUESTION 3

Sketch the graph of $f(x) = ax^3 + bx^2 + cx + d$ if:

- $f'(-3) = f'(5) = 0$
- $f(-3) = 5$ and $f(5) = -3$
- f has three unequal roots.

(Clearly show the coordinates of the turning point and the point of inflection)

QUESTION 4

The following information about a cubic polynomial, $y = f(x)$ is given:

- $f(-1) = 0$
- $f(2) = 0$
- $f(1) = -4$
- $f(0) = -2$
- $f'(1) = f'(-1) = 0$
- if $x < -1$ then $f(x) > 0$
- if $x > 1$ then $f(x) > 0$

4.1 Use the information to draw a neat sketch of f

4.2 For which values of x is f increasing?

4.3 Use your graph to determine the $x -$ coordinate of the point of inflection of f

4.4 For which values of x is f concave up?

Finding the Equation of a Cubic Function

It is important to note that the number of conditions given in the questions must be at least equal to the constants that need to be determined to formulate some equations. Conditions are usually given in the form of coordinates which can either be intercepts, turning points or any other points on the graph of the functions.

Forms of the cubic function:

Standard form: $f(x) = ax^3 + bx^2 + cx + d$

3 intercepts form: $f(x) = a(x - x_1)(x - x_2)(x - x_3)$

Where; x_1, x_2 and x_3 are x −intercept

2 intercepts form: $f(x) = a(x - x_1)(x - x_2)^2$

Where; x_1 is only an x −intercept and x_2 is both an x −intercept and a turning point.

The turning point and the point of contact of tangents provides more than 1 condition. With the turning point or the point of contact of a graph and a tangent, we can easily construct more than 1 equations.

If given coordinates of a turning point $f(x)$ (2 conditions):

i. The pair of coordinates can be substituted in $f(x)$

ii. The x −coordinate can be substituted in $f'(x)$ then equate to zero

If given coordinates of a point of contact of $f(x)$ and its tangent (2 conditions):

i. The pair of coordinates can be substituted in $f(x)$

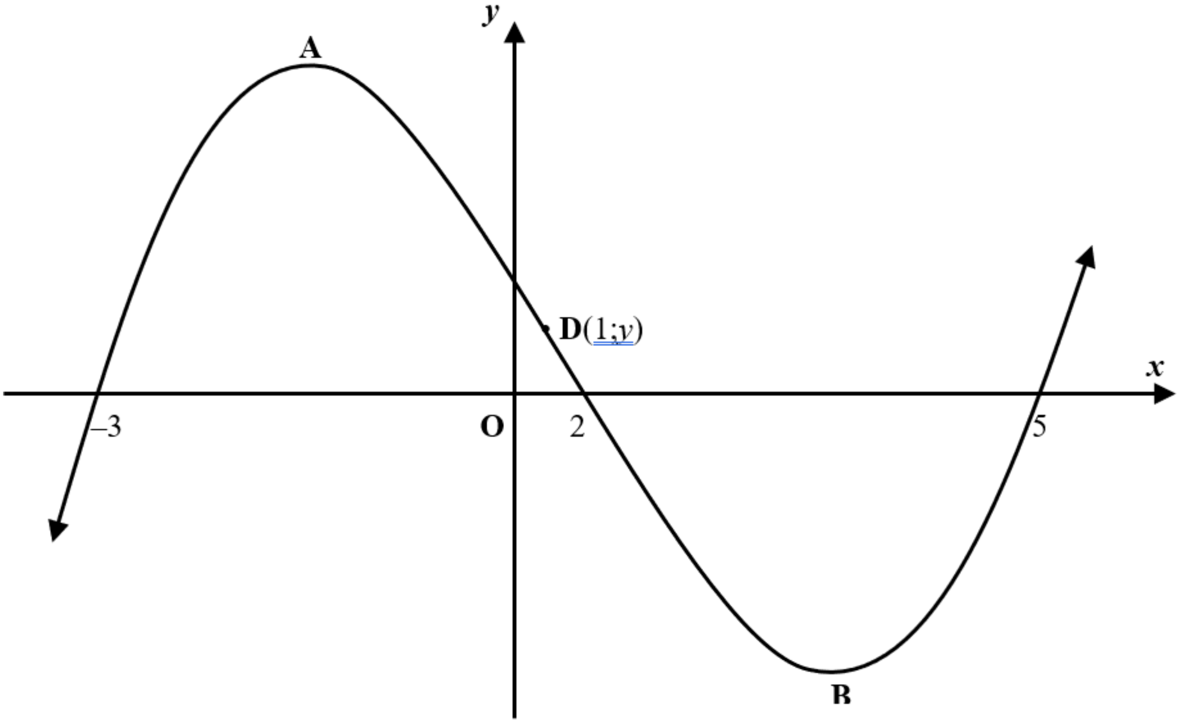
ii. The x −coordinate can be substituted in $f'(x)$ then equate to gradient of tangent at the point of contact.

If given any other coordinates on $f(x)$

i. The pair of coordinates can be substituted in $f(x)$

Examples 1

The graph shows the curve of $f(x) = x^3 + bx^2 + cx + d$. A is a local maximum, and B is a local minimum. The x-intercepts are -3, 2 and 5. $D(1, y)$ is a point on f.



1.1. Show that $a = -4$; $b = -11$ and $c = 30$.

1.2. Hence, write down $f'(x)$.

1.3. Determine the coordinates of the turning point A and B.

1.4. Determine the value of y at point D.

1.5. Find the equation of the tangent to the curve at point D $(1; y)$.

Solutions

1.1

$$f(x) = (x+3)(x-2)(x-5)$$
$$= (x+3)(x^2 - 7x + 10)$$
$$= x^3 - 4x^2 - 11x + 30$$
$$\therefore \quad a = -4 \quad b = -11 \quad c = 30$$

1.2

$$f(x) = x^3 - 4x^2 - 11x + 30$$
$$f^{/}(x) = 3x^2 - 8x - 11$$

1.3 $f(x) = x^3 - 4x^2 - 11x + 30$

$f^{/}(x) = 3x^2 - 8x - 11$

$0 = (3x - 11)(x + 1)$

$\therefore \quad x = \frac{11}{3} \quad \text{or} \quad x = -1$

$A\left(\frac{11}{3}; \frac{400}{27}\right)$ and $B(-1; 36)$

1.4 $f(x) = x^3 - 4x^2 - 11x + 30$

$f(1) = (1)^3 - 4(1)^2 - 11(1) + 30 = 16$

$D(1; 16)$

1.5 $f^{/}(x) = 3x^2 - 8x - 11$

$f^{/}(1) = 3(1)^2 - 8(1) - 11 = -16$

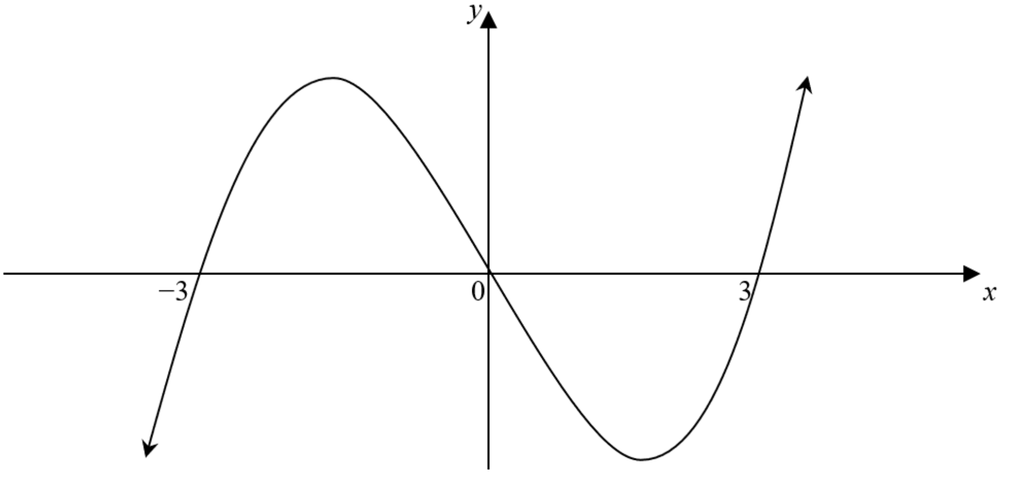
$y - 16 = -16(x - 1)$

$\therefore \quad y = -16x + 32$

Exercise 6

Question 1

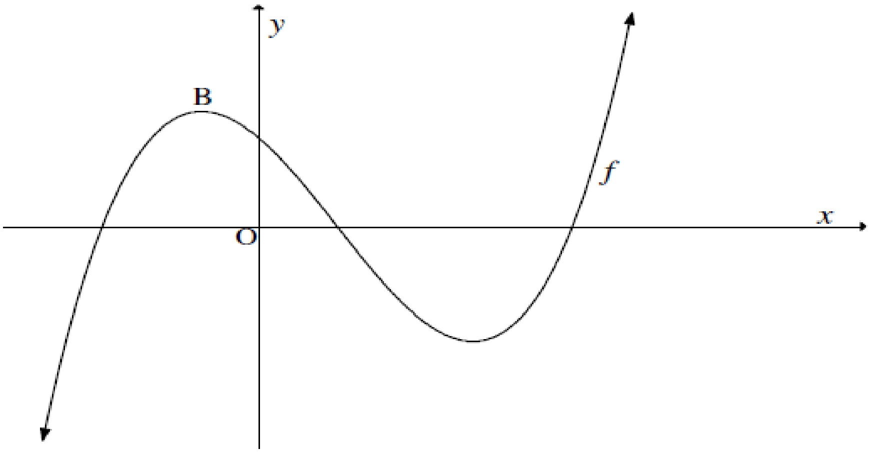
The diagram below shows the graph of $f(x) = x^3 + ax^2 + bx + c$. The intercepts are $(3;\ 0)$, $(0;\ 0)$ and $(-3;\ 0)$



Determine the equation of f.

Question 2

The sketch below represents the curve of $f(x) = x^3 + bx^2 + cx + d$. The solutions of the equation $f(x) = 0$ are -2 ; 1 and 4.



2.1 Calculate the values of a, b and c.

2.2 Calculate the x coordinate of B, the maximum turning point of f.

2.3 Determine the equation of the tangent to the graph of f at $x = -1$.

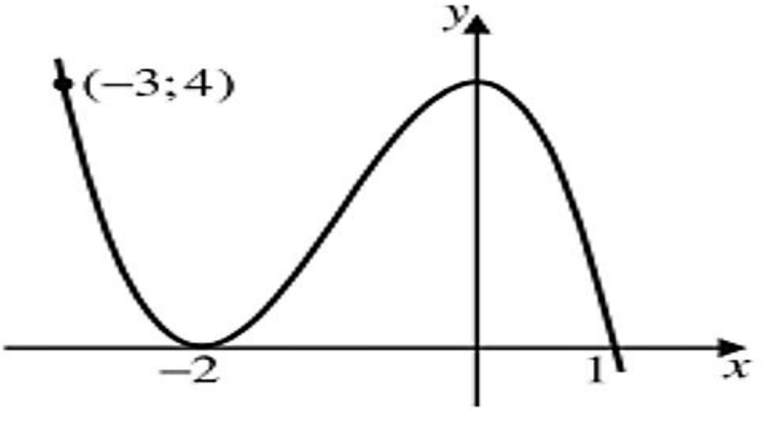
2.4 Sketch the graph of $f^{//}(x)$. Clearly indicate the x- and y-intercepts on your sketch.

2.5 For which value(s) of x is $f(x)$ concave upwards?

Question 3

Determine the equation of the given graph:

$f(x) = ax^3 + bx^2 + cx + d$



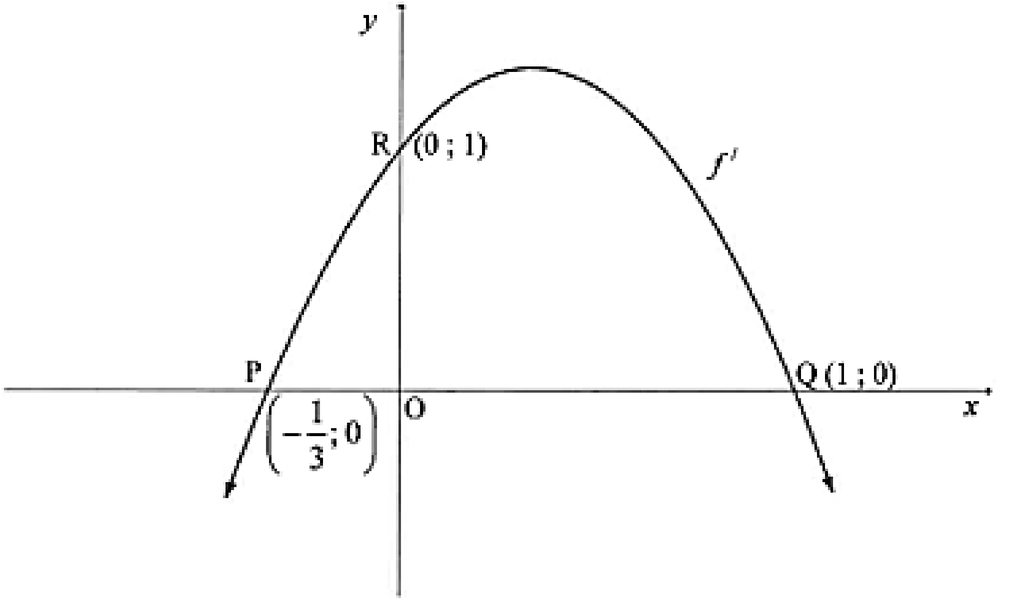
Relating the Key Features of f, f', and f'' to Each Other

1. The value of x-intercept of f'' is the same as the x-value of the coordinates of the turning point of f'. It is also the x-value of the point of inflection of f.
2. The value of x-intercepts of f' are the same as x-values of the coordinates of the turning point of f.

Example 1

The graph of $y = f'(x) = mx^2 + nx + k$ is drawn below.

The graph passes the points $P\left(-\frac{1}{3}\,;0\right)$, $Q(1\,;0)$ and $R(0\,;1)$.



8.1 Determine the values of m, n and k. (6)

8.2 If it is further given that $f(x) = -x^3 + x^2 + x + 2$:

8.2.1 Determine the coordinates of the turning points of f. (3)

8.1 $f'(x) = mx^2 + nx + k$

$f'(x) = m\left(x + \frac{1}{3}\right)(x-1)$

$1 = m\left(0 + \frac{1}{3}\right)(0-1)$

$1 = -\frac{1}{3}m$

$\therefore m = -3$

$f'(x) = -3\left(x + \frac{1}{3}\right)(x-1)$

$f'(x) = -3\left(x^2 - \frac{2}{3}x - \frac{1}{3}\right)$

$f'(x) = -3x^2 + 2x + 1$

$\therefore n = 2$

$\therefore k = 1$

8.2.1 $f(x) = -x^3 + x^2 + x + 2$

$f\left(-\frac{1}{3}\right) = \frac{49}{27} = 1{,}81$

T.P$\left(-\frac{1}{3} ; \frac{49}{27}\right)$

$f(1) = 3$

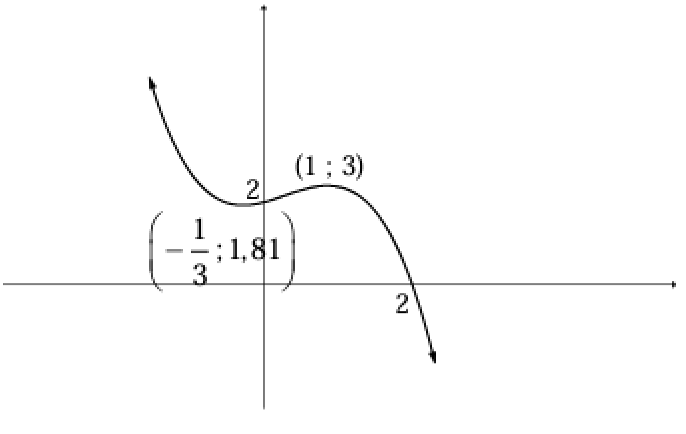
T.P$(1 ; 3)$

8.2.2 $f(x) = -x^3 + x^2 + x + 2$

$-x^3 + x^2 + x + 2 = 0$

$(x-2)(-x^2 - x - 1) = 0$

$x = 2$ or no solution



Optimisation

Applications to Problems Involving Maxima and Minima

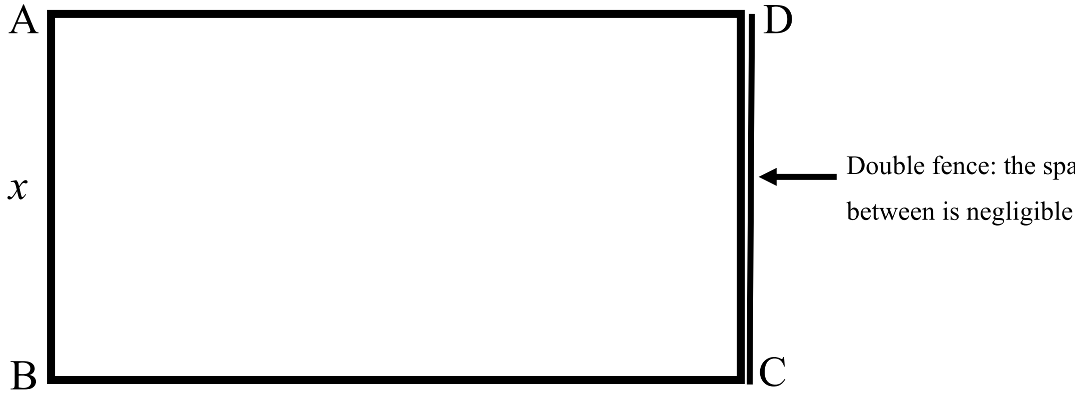
We now focus on some real-world applications of calculus. For example, we can find maximum or minimum area (or volume) as well as minimize variables like cost, perimeter and so forth.

The golden rule for maxima and minima:

In order to maximize or minimize an object $a(x)$, determine $a'(x)$, equate $a'(x)$ to zero and solve for x. The value(s) of x obtained can then be investigated to establish whether they will yield maximum or minimum values of that given object.

Example 1

A farmer makes a rectangular enclosure for his sheep using 100 metres of fencing. He puts a double layer of fencing along one side to stop the sheep breaking through to his crops in the next field. A plan of his fence is shown below. A farmer makes a rectangular enclosure for his sheep using 100 metres of fencing. He puts a double layer of fencing along one side to stop the sheep breaking through to his crops in the next field. A plan of his fence is shown below.



The length of AB is x metres.

1.1 Show that the area of the land which can be enclosed, in this way is given by the formula: $A(x) = 50x - \frac{3}{2}x^2$

1.2 Determine the length of AB if the enclosed area is to be a maximum.

1.1 Length $2BC = 100\text{-}3x$

$$\text{Length } BC = \frac{100 - 3x}{2}$$

$$= 50 - \frac{3}{2}x$$

Area $= AB.BC$

$$A(x) = x\left(50 - \frac{3}{2}x\right)$$

$$= 50x - \frac{3}{2}x^2$$

1.2 $A'(x) = 50 - 3x$

$$0 = 50 - 3x$$

$$\therefore \quad x = \frac{50}{3}m$$

$$AB = \frac{50}{3}m$$

Example 2

A dairy company wishes to market its milk in rectangular containers. The volume of the container must be exactly $1l$, and the length of the base must be twice the breadth of the base. $\left[1l = 1000cm^3\right]$

2.1 Show that the height(h) of the container can be written as $\dfrac{500}{x^2}$, where x is the breadth of the base.

2.2 Determine the dimensions of the container if the surface area must be a maximum, Ignore the thickness of the carton.

Solution

2.1 $V = l \times b \times h$

$$1000cm^3 = 2xcm \times xcm \times hcm$$

$$\therefore h = \frac{1000}{2x^2} = \frac{500}{x^2}$$

2.2 $$SA = 2\left(x \times \frac{500}{x^2}\right) + 2\left(2x \times \frac{500}{x^2}\right) + 2(x \times 2x)$$

$$= \frac{1000}{x} + \frac{2000}{x} + 4x^2$$

$$= 3000x^{-1} + 4x^2$$

$$\therefore \frac{dA}{dx} = -3000x^{-2} + 8x = 0$$

$$8x^3 = 3000$$

$$x^3 = 375$$

$$x = 7{,}21$$

$$\therefore l = 14{,}42cm; b = 7{,}21cm; h = 9{,}62cm$$

Example 3

A Petrol tank at BP Depot has both the inlet and the outlet pipes which are used to control the amount of petrol it contains. The depth of the tank is given by $D(t) = 6 + \frac{t^2}{4} - \frac{t^3}{8}$ where D is in metres and t is in hours that are measured from 9h00.

3.1 Determine the rate at which the depth is changing at 12h00 and then tell whether there is and increase or decrease in depth. Answer correct to two decimal digits.

3.2 At what time will the inflow of petrol be the same as the outflow?

Solution

3.1 $$D(t) = 6 + \frac{t^2}{4} - \frac{t^3}{8}$$

$$D'(t) = \frac{t}{2} - 3\frac{t^2}{8}\checkmark$$

$$D'(3) = \frac{3}{2} - \frac{3(3)^{2\checkmark}}{8} = -\frac{15}{8} m.h^{-1}\checkmark$$

$= -1{,}875\checkmark$ decreasing in depth

3.2 $$D'(t) = 0$$

$$\frac{t}{2} - 3\frac{t^2}{8} = 0$$

$$t\left(\frac{1}{2} - 3\frac{t}{8}\right) = 0$$

$t = 0$ or $t = \frac{4}{3}$

At 9h00 and at $\frac{4}{3} \times 60 = 80$ minutes later. i.e. at 10h20 *9h00* $\frac{4}{3} \times 60 = 80$ *minute later 10h20*

Distance (Displacement), Speed (Velocity) and Acceleration/ Deceleration

- $s(t)$ represents an equation of motion (**height, distance, displacement**) at time *t*.
- $s'(t)$ represents **velocity or speed** at time *t*.
- $s''(t)$ represents **acceleration** at time *t*.

Example 4

A ball is thrown into the air and its height, *h*, above the ground, after *t* seconds is given by:

$h(t) = -5t^2 + 25t + 4$ metres.

Determine:

a) The height of the ball above the ground after 2 seconds

b) The velocity of the ball at that moment.

c) The maximum height of the ball above the ground.

d) The time taken to reach a downwards velocity of $5\,m/s$.

Solution

(a)
$$h(t) = -5t^2 + 25t + 4$$
$$h(2) = -5(2)^2 + 10(2) + 25$$
$$= 34m$$

(b)
$$h(t) = -5t^2 + 25t + 4$$
$$h'(t) = -10t + 25$$
$$h'(2) = -10(2) + 25$$
$$= m/s$$

(c)
$$h'(t) = -10t + 25$$
$$0 = -10t + 25$$
$$\therefore \quad t = 2,5s$$
$$\text{Max height} = -5(2,5)^2 + 25(2,5) + 4$$
$$= 35,25m$$

(d)
$$h'(t) = -10t + 25$$
$$5 = -10t + 25$$
$$10t = 20$$
$$\therefore \; t = 2s$$

Therefore, the time taken to reach a downward velocity of 5*m/s* is 4,5*s*

Exercise 7

Question 1

1. A water tank with an inlet and an outlet is used to water the garden. The equation $D(t) = 3 + \frac{1}{2}t^2 - \frac{1}{4}t^3$ gives the depth of water in metres where t is the time in hours that has elapsed since 07:30.

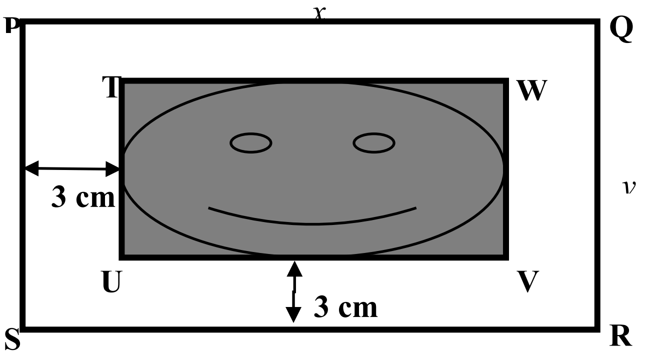
   1.1 What is the depth of the water at 09:30?

   1.2 At what rate does the depth of the water change at 10:30?

   1.3 At what time will the inflow of the water be the same as the outflow of the water?

Question 2

In the diagram below, TUVW is a rectangular picture. The picture is framed such that there is a 3 cm space around the picture. The perimeter of the rectangle PQRS is 70 cm.
PQ = x units and QR = y units.



Calculate the maximum area of the picture TUVW.

Question 3

3 A dairy company wishes to market its milk in rectangular carton containers. The volume of the container must be exactly $1l$, and the length of the base must be twice the breadth of the base. [$1l = 1000cm^3$]

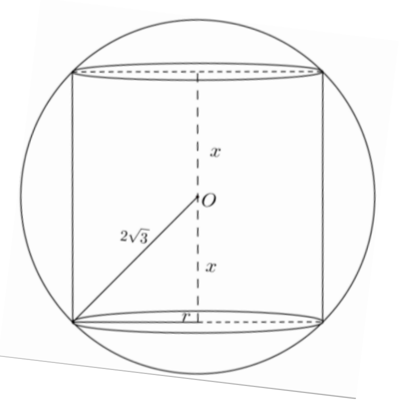
   3.1 Show that the height (h) of the container can be written as $\frac{500}{x^2}$, where x is the breadth of the base

   3.2 Determine the dimensions of the container if the surface area must be a minimum.

   Ignore the thickness of the carton

Question 4

4 The diagram below shows a cylindrical can that fits into circle O with radius $R = 2\sqrt{3}$ units.



4.1 Express r in terms of x.

4.2 Calculate the value of x for which the volume of the cylinder is a maximum

4.3 Calculate the maximum volume of the can in terms of π.

Question 5

1.5 The height to which the plant grows during the first six months is given by the following function: $f(x) = 36x - 3x^2$; $0 \leq x \leq 6$ where x is the age of the plant in months and $f(x)$ the height in centimetres above the ground after x months.

1.5.1 What height would the plant reach at the end of 3 months?

1.5.2 At the end of how many months will the plant reach its maximum height?

1.5.3 Hence calculate the maximum height to which the plant will grow.

EUCLIDEAN GEOMETRY

Examination Guideline

Acceptable Reasons

THEOREM STATEMENT	ACCEPTABLE REASON(S)
LINES	
The adjacent angles on a straight line are supplementary.	$\angle$s on a str line
If the adjacent angles are supplementary, the outer arms of these angles form a straight line.	adj $\angle$s supp
The adjacent angles in a revolution add up to 360°.	$\angle$s round a pt **OR** $\angle$s in a rev
Vertically opposite angles are equal.	vert opp $\angle$s =
If AB || CD, then the alternate angles are equal.	alt $\angle$s; AB || CD
If AB || CD, then the corresponding angles are equal.	corresp $\angle$s; AB || CD
If AB || CD, then the co-interior angles are supplementary.	co-int $\angle$s; AB || CD
If the alternate angles between two lines are equal, then the lines are parallel.	alt $\angle$s =
If the corresponding angles between two lines are equal, then the lines are parallel.	corresp $\angle$s =
If the cointerior angles between two lines are supplementary, then the lines are parallel.	coint $\angle$s supp

TRIANGLES	
The interior angles of a triangle are supplementary.	$\angle$ sum in Δ **OR** sum of $\angle$s in Δ **OR** Int $\angle$s Δ
The exterior angle of a triangle is equal to the sum of the interior opposite angles.	ext $\angle$ of Δ
The angles opposite the equal sides in an isosceles triangle are equal.	$\angle$s opp equal sides
The sides opposite the equal angles in an isosceles triangle are equal.	sides opp equal $\angle$s
In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.	Pythagoras **OR** Theorem of Pythagoras
If the square of the longest side in a triangle is equal to the sum of the squares of the other two sides then the triangle is right-angled.	Converse Pythagoras **OR** Converse Theorem of Pythagoras
If three sides of one triangle are respectively equal to three sides of another triangle, the triangles are congruent.	SSS
If two sides and an included angle of one triangle are respectively equal to two sides and an included angle of another triangle, the triangles are congruent.	SAS **OR** S$\angle$S
If two angles and one side of one triangle are respectively equal to two angles and the corresponding side in another triangle, the triangles are congruent.	AAS **OR** $\angle\angle$S
If in two right angled triangles, the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of the other, the triangles are congruent	RHS **OR** 90°HS

THEOREM STATEMENT	ACCEPTABLE REASON(S)
The line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side	Midpt Theorem
The line drawn from the midpoint of one side of a triangle, parallel to another side, bisects the third side.	line through midpt ‖ to 2nd side
A line drawn parallel to one side of a triangle divides the other two sides proportionally.	line ‖ one side of Δ **OR** prop theorem; name ‖ lines
If a line divides two sides of a triangle in the same proportion, then the line is parallel to the third side.	line divides two sides of Δ in prop
If two triangles are equiangular, then the corresponding sides are in proportion (and consequently the triangles are similar).	||| Δs **OR** equiangular Δs
If the corresponding sides of two triangles are proportional, then the triangles are equiangular (and consequently the triangles are similar).	Sides of Δ in prop
If triangles (or parallelograms) are on the same base (or on bases of equal length) and between the same parallel lines, then the triangles (or parallelograms) have equal areas.	same base; same height **OR** equal bases; equal height

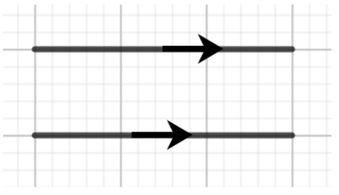
THEOREM STATEMENT	ACCEPTABLE REASON(S)
Equal chords in equal circles subtend equal angles at the circumference of the circles.	equal circles; equal chords; equal ∠s
Equal chords in equal circles subtend equal angles at the centre of the circles.	equal circles; equal chords; equal ∠s
The opposite angles of a cyclic quadrilateral are supplementary	opp ∠s of cyclic quad
If the opposite angles of a quadrilateral are supplementary then the quadrilateral is cyclic.	opp ∠s quad supp **OR** converse opp ∠s of cyclic quad
The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.	ext ∠ of cyclic quad
If the exterior angle of a quadrilateral is equal to the interior opposite angle of the quadrilateral, then the quadrilateral is cyclic.	ext ∠ = int opp ∠ **OR** converse ext ∠ of cyclic quad
Two tangents drawn to a circle from the same point outside the circle are equal in length	Tans from common pt **OR** Tans from same pt
The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.	tan chord theorem
If a line is drawn through the end-point of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.	converse tan chord theorem **OR** ∠ between line and chord

QUADRILATERALS	
The interior angles of a quadrilateral add up to 360°.	sum of $\angle$s in quad
The opposite sides of a parallelogram are parallel.	opp sides of $\|$m
If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.	opp sides of quad are $\|$
The opposite sides of a parallelogram are equal in length.	opp sides of $\|$m
If the opposite sides of a quadrilateral are equal , then the quadrilateral is a parallelogram.	opp sides of quad are = **OR** converse opp sides of a parm
The opposite angles of a parallelogram are equal.	opp $\angle$s of $\|$m
If the opposite angles of a quadrilateral are equal then the quadrilateral is a parallelogram.	opp $\angle$s of quad are = **OR** converse opp angles of a parm
The diagonals of a parallelogram bisect each other.	diag of $\|$m
If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.	diags of quad bisect each other **OR** converse diags of a parm
If one pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.	pair of opp sides = and $\|$
The diagonals of a parallelogram bisect its area.	diag bisect area of $\|$m
The diagonals of a rhombus bisect at right angles.	diags of rhombus
The diagonals of a rhombus bisect the interior angles.	diags of rhombus
All four sides of a rhombus are equal in length.	sides of rhombus
All four sides of a square are equal in length.	sides of square
The diagonals of a rectangle are equal in length.	diags of rect
The diagonals of a kite intersect at right-angles.	diags of kite
A diagonal of a kite bisects the other diagonal.	diag of kite
A diagonal of a kite bisects the opposite angles	diag of kite

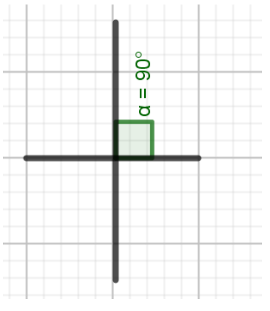
Lines, Angles and Triangles

Lines and Angles

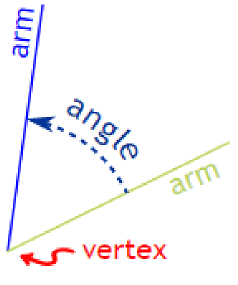
Parallel lines → straight lines on the same plane and that can never intersect as they are extended



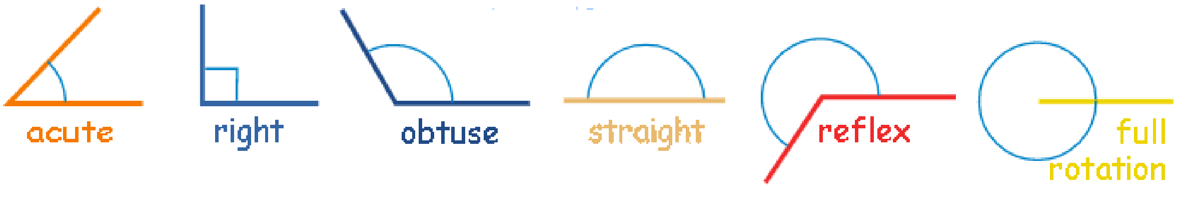
Perpendicular lines → straight lines that intersect at right angles (90˚)



Angles: an angle is "the amount of turn between two lines around their common point (the vertex)"



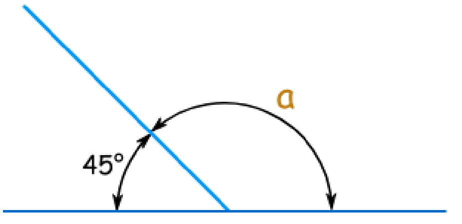
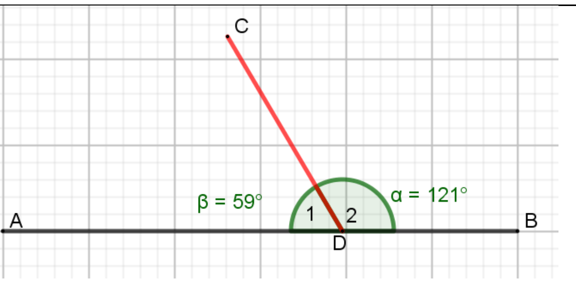
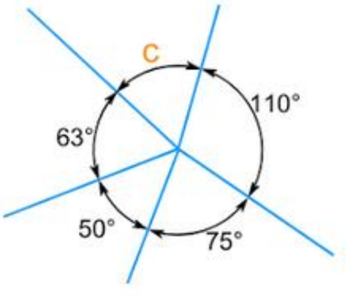
Types of angles

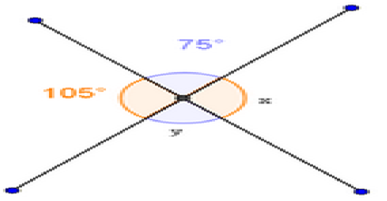
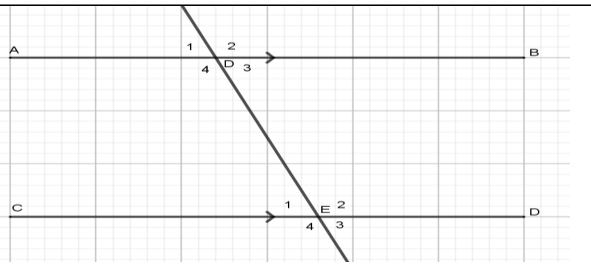
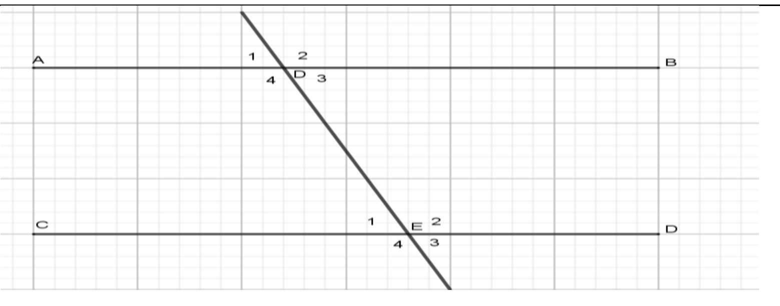
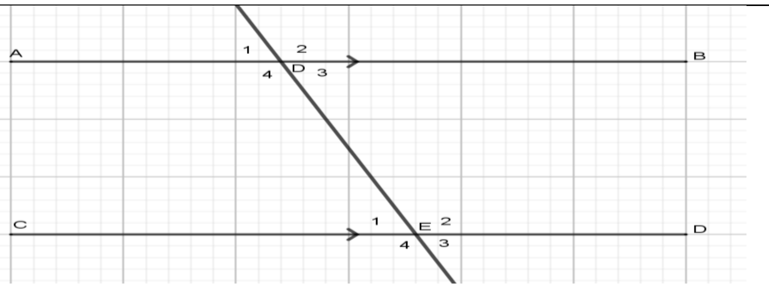


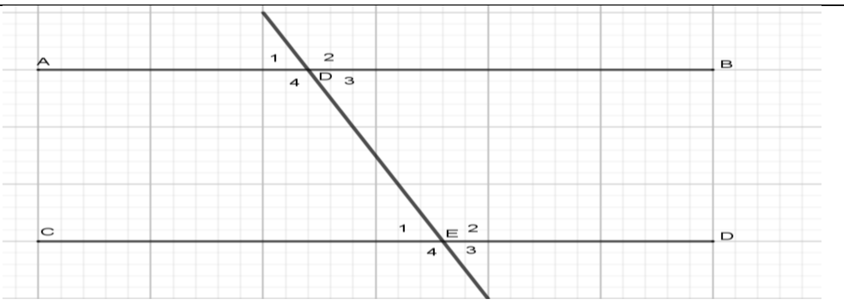
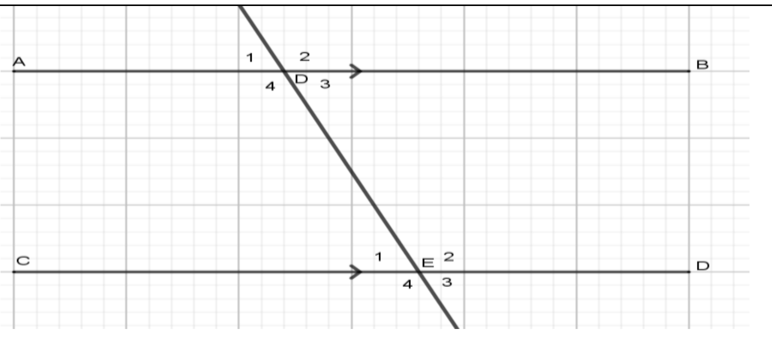
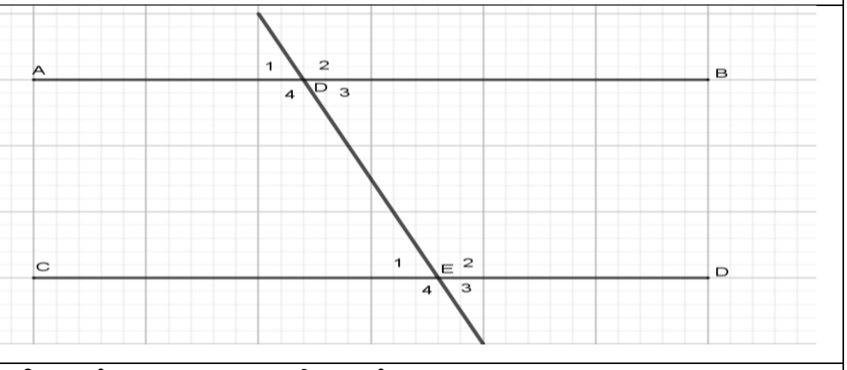
- Acute angle → an angle between 0° and 90°
- Right angle → an angle equal 90°
- Obtuse angle → an angle between 90° and 180°
- Straight angle → an angle equal 180°
- Reflex angle → an angle between 180° and 360°
- Full rotation (Revolution) → an angle equal 360°

(www.mathisfun.com)

N.B All the theorem statements were taken from 2021 grade 12 mathematics examination guidelines

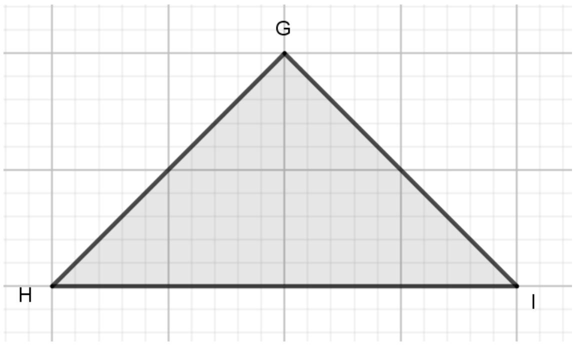
1.	Theorem statement	The adjacent angles on a straight line are supplementary
	Diagram	
	Mathematical statement	$\boldsymbol{a + 45° = 180°}$ $\boldsymbol{a = 180° - 45°}$ $\boldsymbol{a = 135°}$
	Reason	$\angle s$ on a str line
2	Theorem statement	If the adjacent angles are supplementary, the outer arms of these angles form a straight line
	Diagram	
	Mathematical statement	$\hat{D}_1 + \hat{D}_2 = 59° + 121° = 180°$
	Reason	adj $\angle s$ supp
3	Theorem statement	The adjacent angles in a revolution add up to 360°
	Diagram	
	Mathematical statement	$\boldsymbol{\hat{C} + 110° + 75° + 50° + 63° = 360°}$ $\boldsymbol{\hat{C} = 62°}$
	Reason	$\angle s$ around a pt

4.	Theorem statement	Vertically opposite angles are equal
	Diagram	
	Mathematical statement	$x = 105°$ **and** $y = 75°$
	Reason	vert opp ∠s =
5	Theorem statement	If AB ‖ CD, then the alternate angles are equal
	Diagram	
	Mathematical statement	$\hat{D}_3 = \hat{E}_1$ and $\hat{D}_4 = \hat{E}_2$
	Reason	alt ∠s ; AB || CD
6	Theorem statement	If the alternate angles between two lines are equal, then the lines are parallel
	Diagram	
	Mathematical statement	If $\hat{D}_3 = \hat{E}_1$ and $\hat{D}_4 = \hat{E}_2$ then AB ‖ CD
	Reason	alt ∠s =
7.	Theorem statement	If AB ‖ CD, then the corresponding angles are equal
	Diagram	
	Mathematical statement	$\hat{D}_3 = \hat{E}_3$, $\hat{D}_4 = \hat{E}_4$, $\hat{D}_1 = \hat{E}_1$ and $\hat{D}_2 = \hat{E}_2$

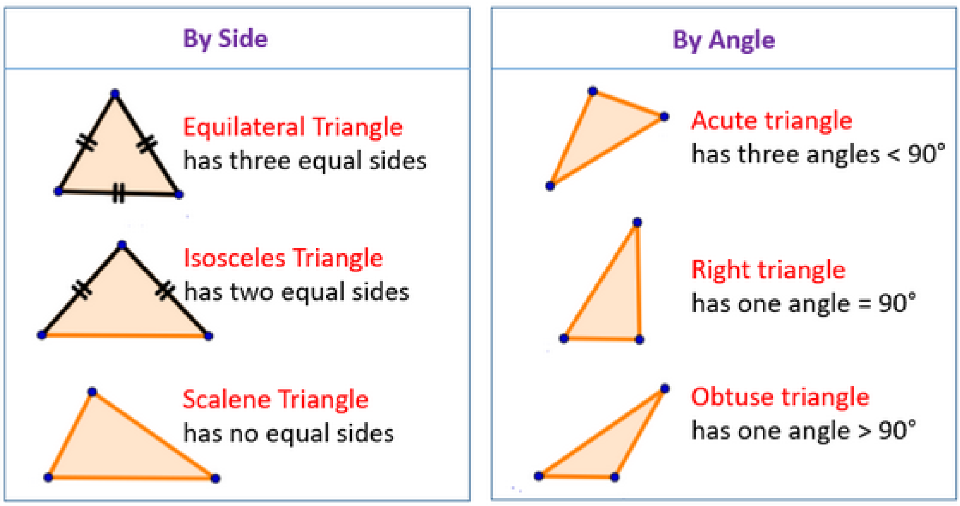
	Reason	corresp ∠s ; AB || CD
8	Theorem statement	If the corresponding angles between two lines are equal, then the lines are parallel
	Diagram	
	Mathematical statement	If $\hat{D}_3=\hat{E}_3$, $\hat{D}_4=\hat{E}_4$, $\hat{D}_1=\hat{E}_1$ and $\hat{D}_2=\hat{E}_2$ then AB || CD
	Reason	corresp ∠s =
9.	Theorem statement	If AB || CD, then the co-interior angles are supplementary
	Diagram	
	Mathematical statement	$\widehat{D}_3+\hat{E}_2=180^0$ and $\widehat{D}_4+\hat{E}_1=180^0$
	Reason	co-int ∠s ; AB || CD
10	Theorem statement	If the co-interior angles between two lines are supplementary, then the lines are parallel
	Diagram	
	Mathematical statement	If $\widehat{D}_3+\hat{E}_2=180^0$ and $\widehat{D}_4+\hat{E}_1=180^0$ then AB || CD
	Reason	co-int ∠s supp

Triangles

Triangles: a triangle is shape made of 3 straight lines; the shape is closed. It has 3 sides and 3 vertices.



Types of triangles



(www.curemath.com)

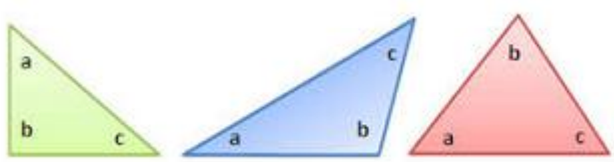
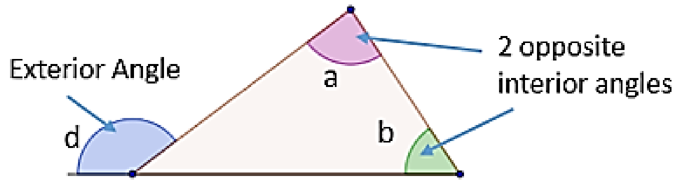
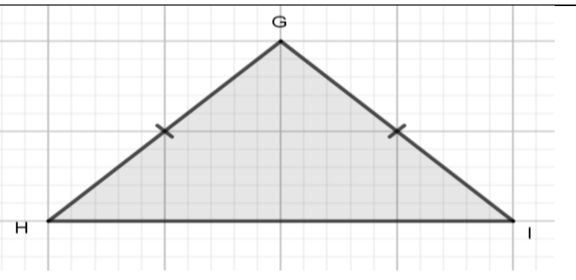
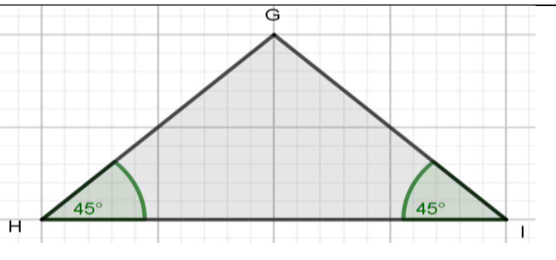
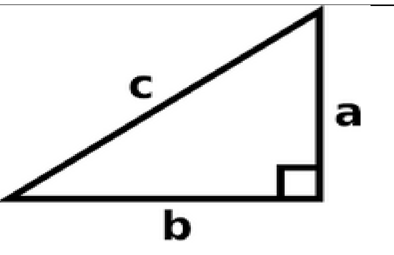
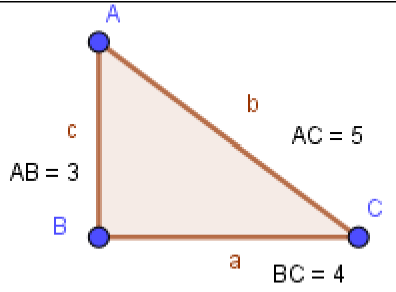
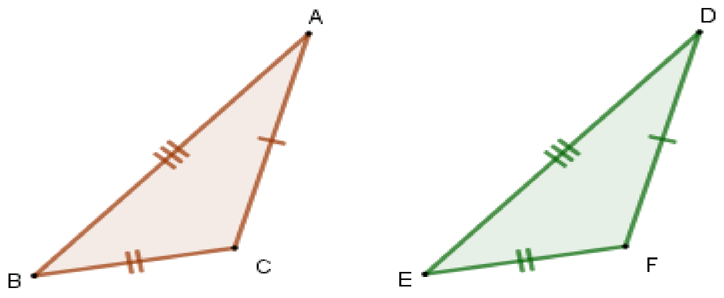
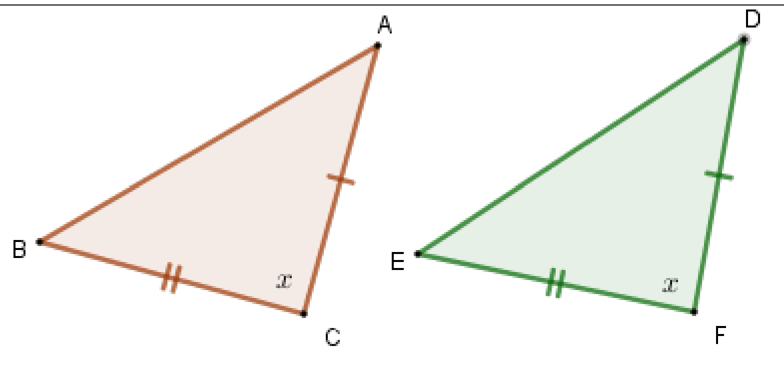
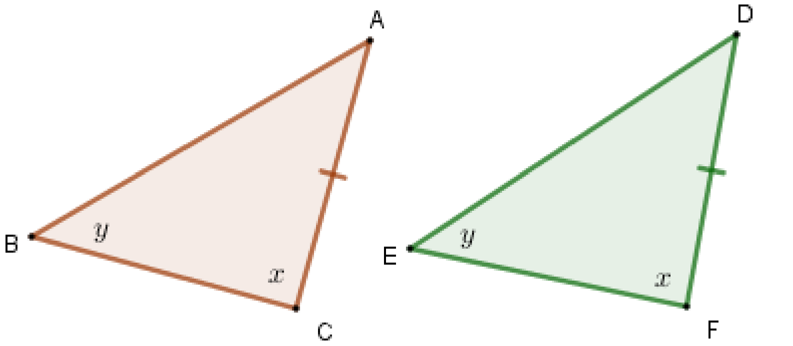
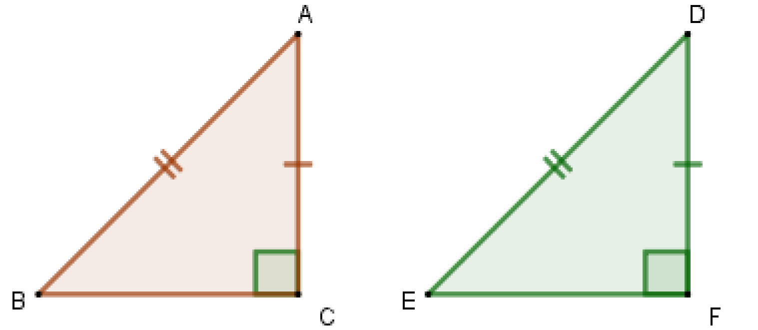
1.	Theorem statement	The interior angles of a triangle are supplementary
	Diagram	
	Mathematical statement	$a + b + c = 180°$
	Reason	sum of ∠s in Δ
2	Theorem statement	The exterior angle of a triangle is equal to the sum of the interior opposite angles

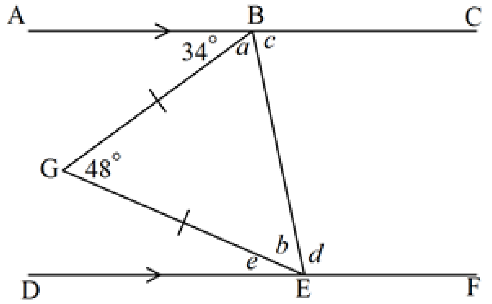
	Diagram	
	Mathematical statement	d = a + b
	Reason	ext ∠ of Δ
3	Theorem statement	The angles opposite the equal sides in an isosceles triangle are equal
	Diagram	
	Mathematical statement	If GH = GI, then $\widehat{H} = \hat{I}$
	Reason	$\angle s$ opp equal sides
4.	Theorem statement	The sides opposite the equal angles in an isosceles triangle are equal
	Diagram	
	Mathematical statement	If $\widehat{H} = \hat{I}$, then HG = GI
	Reason	sides opp equal $\angle s$
5	Theorem statement	In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides
	Diagram	
	Mathematical statement	$c^2 = a^2 + b^2$
	Reason	Pythagoras theorem

6	Theorem statement	If the square of the longest side in a triangle is equal to the sum of the squares of the other two sides then the triangle is right-angled
	Diagram	
	Mathematical statement	If $b^2 = a^2 + c^2$, the ΔABC is right-angled
	Reason	Converse Pythagoras
	Conditions for congruency	
7.	Theorem statement	If three sides of one triangle are respectively equal to three sides of another triangle, the triangles are congruent
	Diagram	
	Mathematical statement	If AB = DE, AC = DF and BC= EF, then $\Delta ABC \equiv \Delta DEF$
	Reason	SSS
8	Theorem statement	If two sides and an included angle of one triangle are respectively equal to two sides and an included angle of another triangle, the triangles are congruent
	Diagram	
	Mathematical statement	If $\hat{C} = \hat{F}$, AC=DF and BC=EF, then $\Delta ABC \equiv \Delta DEF$
	Reason	SAS

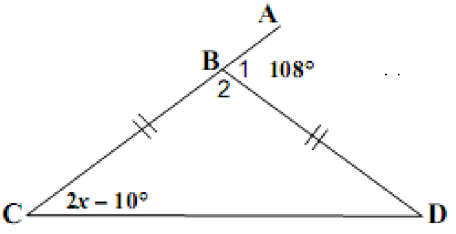
9	Theorem statement	If two angles and one side of one triangle are respectively equal to two angles and the corresponding side in another triangle, the triangles are congruent
	Diagram	
	Mathematical statement	If $\hat{C}$ = F̂ , $\hat{B}$ = Ê and AC=DF, then ΔABC ≡ ΔDEF
	Reason	AAS
10.	Theorem statement	If in two right-angled triangles, the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of the other, the triangles are congruent
	Diagram	
	Mathematical statement	If AC=DF and AB=DE, then ΔABC ≡ ΔDEF
	Reason	RHS

Exercise 1

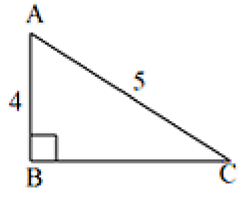
1. Find the size of the angles a, b, c, d and e



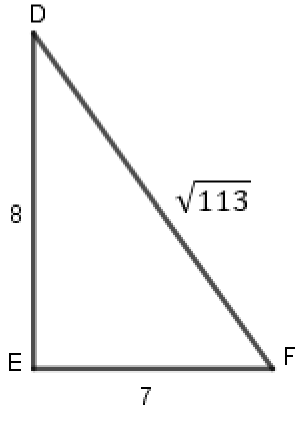
2. Calculate x, if $\hat{B}_1 = 108°$ and $\hat{B}_2 = 2x - 10°$



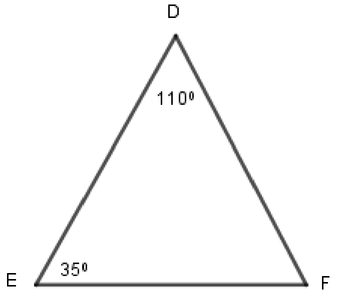
3. Calculate the length of BC



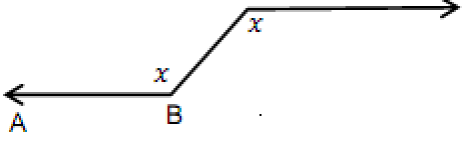
4. Prove that triangle ΔDEF is a right-angled triangle. Show which angle is = 90^o



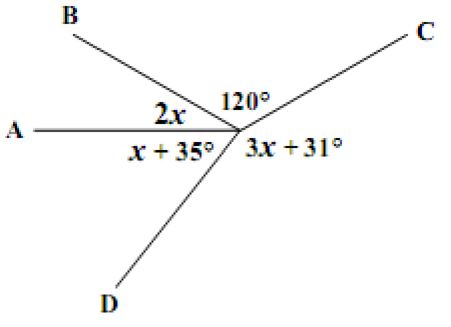
5. Prove that DE = DF



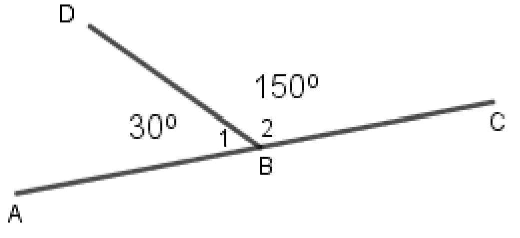
6. B and C are alternating angles equal to x. Is AB || CD? Give a reason for your answer.



7. Calculate x



8. Given that: $A\hat{B}D = 30^o$ and $D\hat{B}C = 150^o$. Prove, giving reasons, that ABC is a straight line

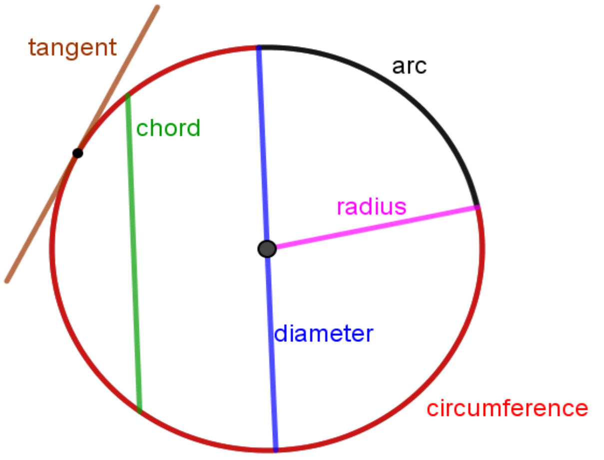


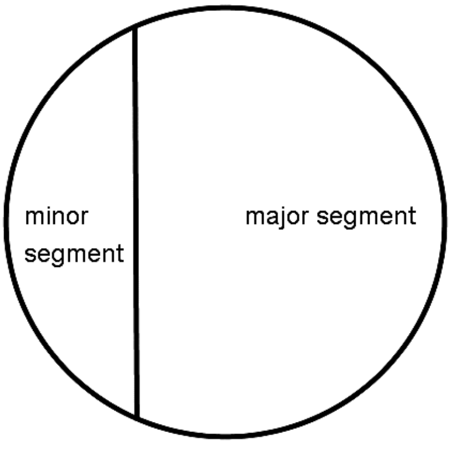
9. 9.1 If the corresponding angles between two lines are equal, then the lines are ………………….

9.2 If the ………………..angles between two lines are supplementary, then the lines are parallel

Circle Geometry

Different parts of a circle:





Circle theorems are divided into four groups:

CENTER THEOREMS	TANGENT THEOREMS	CYCLIC QUAD THEORMES	EQUAL CHORDS COROLLARY
1. The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle. 2. The angle subtended by the diameter at the circumference of the circle is 90 3. The line drawn from the Center of the circle perpendicular to the chord bisects the chord.	1. The angle between the tangent to a circle and a chord drawn from the point of contact is equal to the angle in the alternate segment. 2. Two Tangents drawn to a circle from the same point outside the circle are equal in length . 3. The tangent to a circle is perpendicular to the radius/diameter of the circle at the point of contact.	1. The opposite angles of a cyclic quadrilateral are supplementary. 2. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. 3. Angles subtended by a chord/segment of a circle, on the same side of the chord/segment are equal.	1. Equal chords subtend equal angles at the circumference of the circle. 2. Equal chords subtend equal angles at the center of the circle.

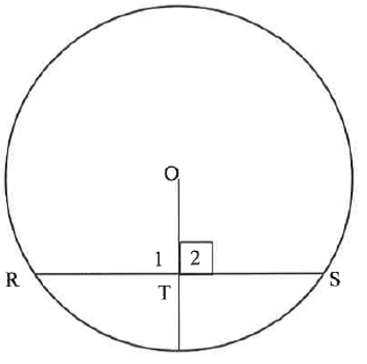
Classical method: This is done by writing two columns (HOLY-CROSS), the first being a list of statements and the second column a matching list of legal justifications. i.e. theorems and axioms referred to as REASONS.

STATEMENT	**REASON**
1. $\hat{A}_2 = \hat{C}_1 = 35°$	Alt $\angle's$, $AB \parallel CD$
2.$M\hat{A}P + 35° = 75°$	Ext $\angle$ of Δ

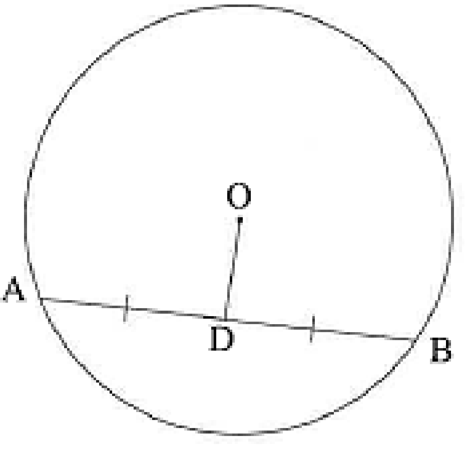
Examinable Proofs

Five proofs of theorems that are examinable from grade 11.

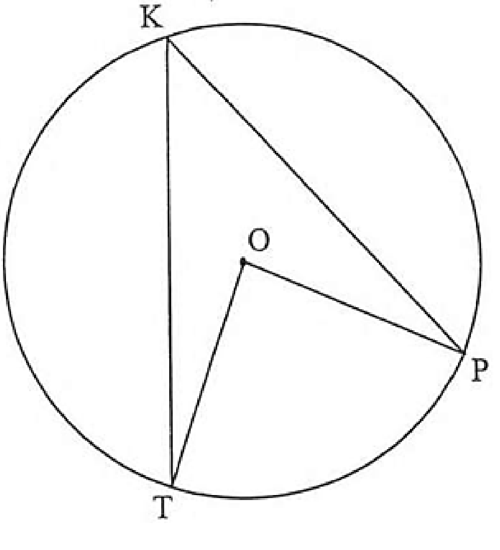
A. In the diagram below, O is the centre of the circle and point T lies on chord RS. Prove the theorem which states that if $OT \perp RTS$ then RT=TS.



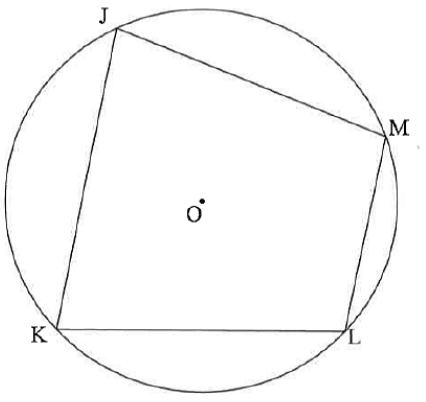
B. In the diagram, O is the centre of a circle. OD bisects AB. Prove the theorem that states that the line from the centre of a circle that bisects a chord is perpendicular to the chord, i.e. $OD \perp AB$.



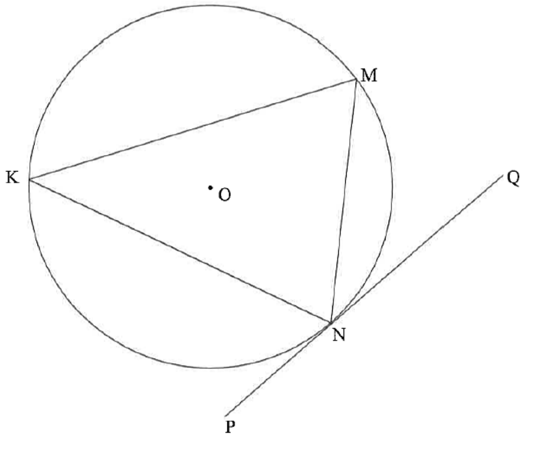
C. In the diagram below, O is the centre of the circle. Use the diagram to prove the theorem which states prove that $T\hat{O}P = 2T\hat{K}P$.



D. In the diagram, JKLM is a cyclic quadrilateral and the circle has centre O. Prove the theorem which states that $\hat{J} + \hat{L} = 180°$



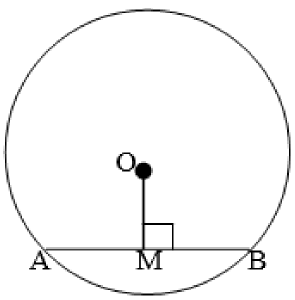
E. In the diagram below, chords KM, MN are drawn in the circle with centre O. PNQ is the tangent to the circle at N. Prove that $M\hat{N}Q = \hat{K}$



Theorems about the Center

The line drawn from centre of the circle perpendicular to a chord bisects the chord.

<u>(line from centre ⊥ to chord)</u>

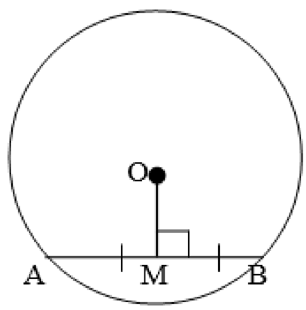


If $OM \perp AB$, then

$AM = MB$

The line segment joining the centre of the circle to the midpoint of a chord is perpendicular to the chord.

<u>(line from centre to midpoint of chord)</u>

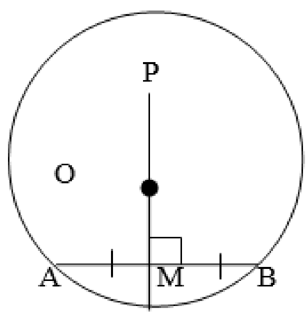


If $AM = MB$, then

$OM \perp AB$

The perpendicular bisector of a chord passes through the centre of the circle.

<u>(perp bisector of chord)</u>

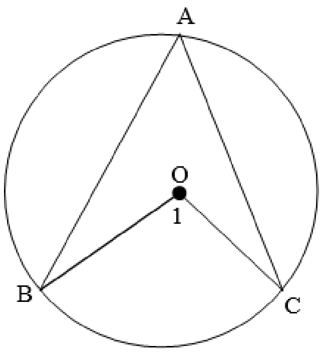


If $OP \perp AB$ and $AM = MB$,

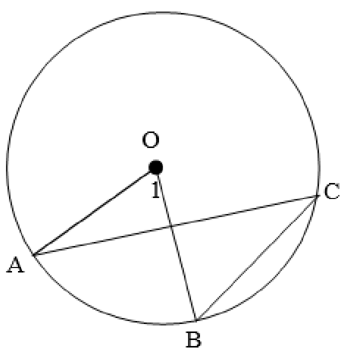
then PM passes through O

The angle which an arc of a circle subtends at the center of the circle is twice the angle it subtends at the circumference of the circle.

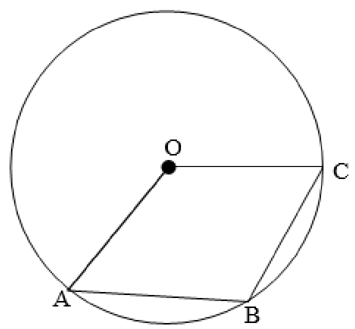
<u>(∠at centre = 2 × ∠ at circle)</u>



$\hat{O}_1 = 2\ \hat{A}$



$\hat{O}_1 = 2\ \hat{C}$

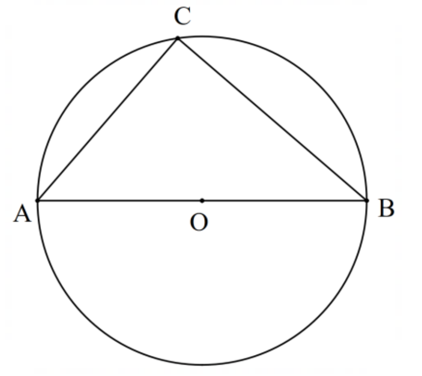


Reflex $A\hat{O}C = 2\ \hat{B}$

JENN MAREMATLOU
A Division of Marematlou Group Holdings

The angle subtended at the circle by the diameter is a right angle.

(∠s in semi-circle) OR (diameter subtends right angle)

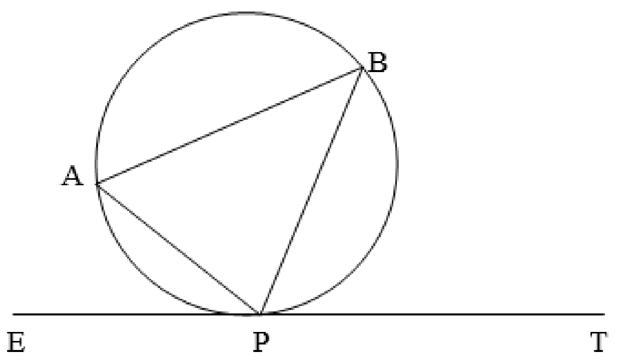


$\hat{C} = 90°$ (angle in a semi-circle)

Theorems about the Tangent

A. The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.

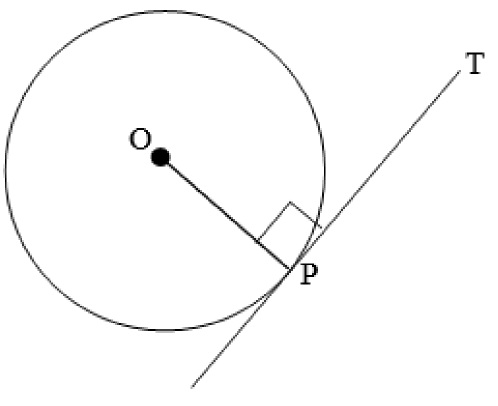
(tan-chord theorem)



$B\hat{P}T = \hat{A}$ and $A\hat{P}E = \hat{B}$

B. A tangent to a circle is perpendicular to the radius at the point of contact.

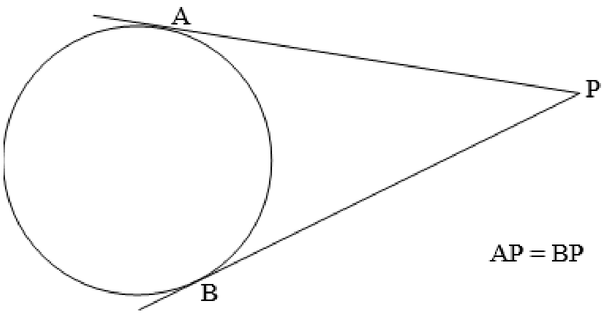
(tan ⊥ radius)



$PT \perp OP$

Two tangents drawn to a circle from the same point outside the circle are equal in length

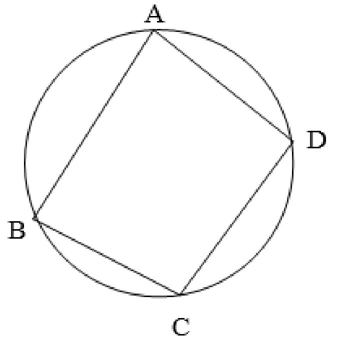
(tans from same pt.)



Theorems about the Cyclic Quad

A. The opposite angles of a cyclic quadrilateral are supplementary.

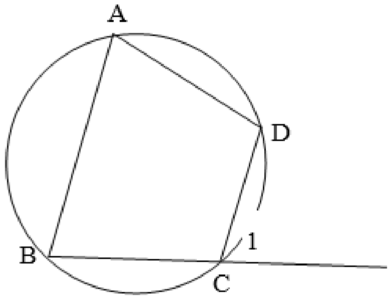
<u>(opp ∠s of cyclic quad)</u>



$\hat{A} + \hat{C} = 180°$ and $\hat{B} + \hat{D} = 180°$

B. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

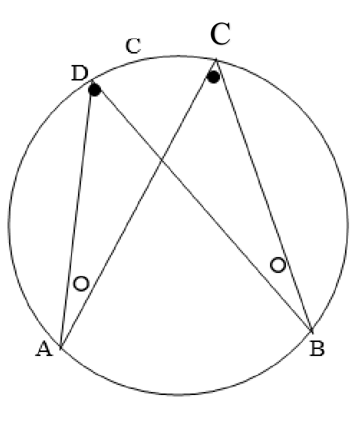
<u>(ext. ∠ of cyclic quad.)</u>



$\hat{C}_1 = \hat{A}$

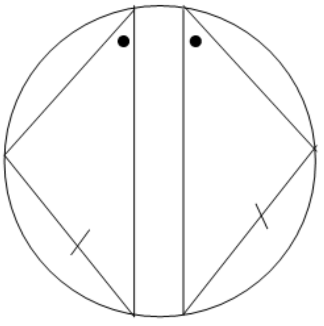
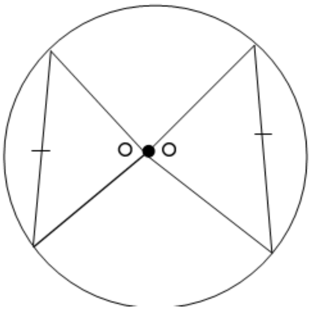
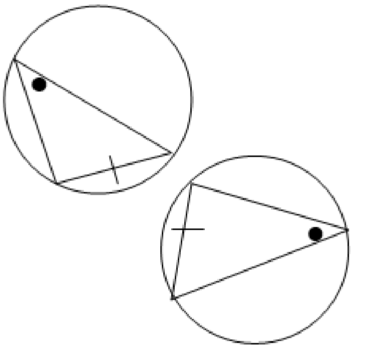
C. Angles subtended by a chord of the circle, on the same side of the chord, are equal.

<u>(∠'s in the same seg)</u>



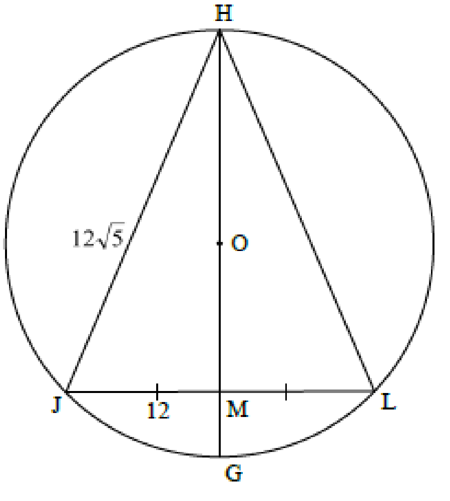
$\hat{D} = \hat{C}$ and $\hat{A} = \hat{B}$

Equal Chords Corollaries

Equal chords subtend equal angles at the circumference of the circle.	Equal chords subtend equal angles at the centre of the circle	Equal chords, in equal circles subtend equal angles.
		

Exercise 2

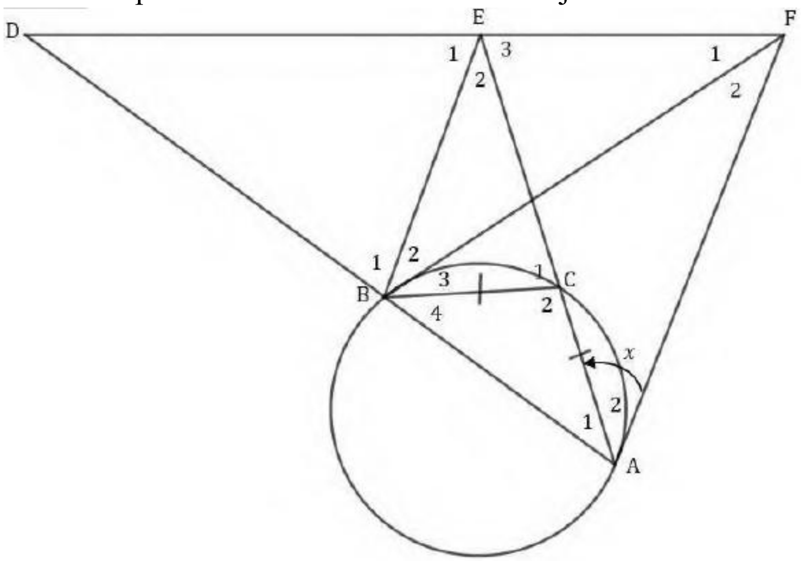
1. In the diagram below, a circle centered at O is drawn. H, J, G and L are points on the circle. ΔHJL is drawn. HOG bisects JL at M. HJ =$12\sqrt{5}$ units and JM = 12 units.



1.1. If MG = 6 units and OM = x, write HM in terms of x.

1.2. Calculate, giving reasons, the length of the radius of the circle.

2. FA and FB are tangents to the circle ABC with $BC = AC$. $FD \parallel CB$ and $C\hat{A}F = x$. Chord AB is produced to D and chord AC is produced to meet DF at E. BC is joined.



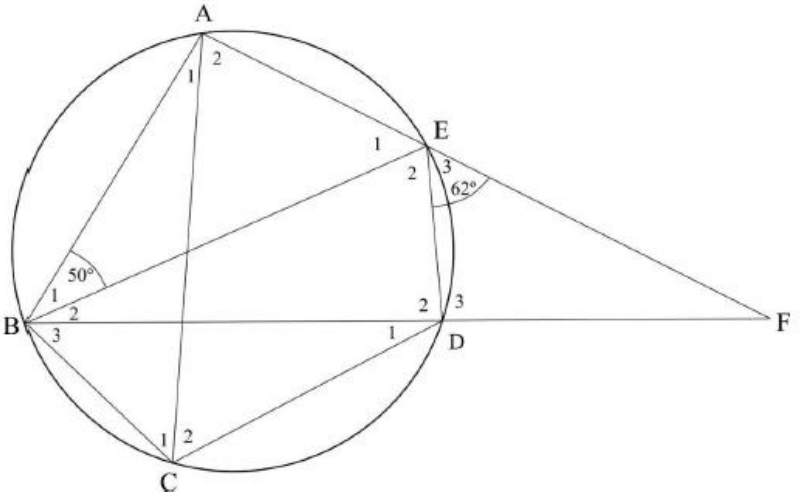
2.1. Write down FIVE other angles equal to x.

2.2. Hence deduce that:

2.2.1 ABEF is a cyclic quadrilateral

2.2.2 $AF = BD$

3. In the given diagram, A, B , C ,D and E are points on the circle. BE is a diameter. $\widehat{E_3} = 62°$ and $\widehat{B_1} = 50°$. BD produced meets AE produced at F.



3.1. Determine, with reasons :

3.1.1. $B\hat{A}E$

3.1.2. $\widehat{E_1}$

3.1.3. $\widehat{C_1}$

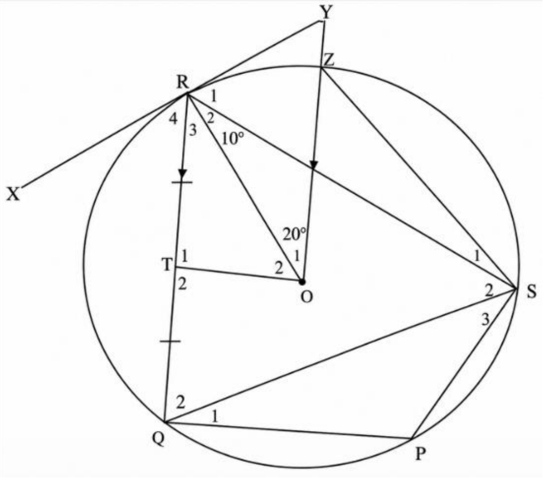
3.1.4. $\widehat{C_2}$

3.1.5. $A\hat{B}D$

3.2. Complete the following theorem statement:

The angle between the tangent to a circle and the chord drawn from the point of contact is …

4. In the diagram below, points P, Q, R and S are points on a circle with centre O. OT bisects chord QR at T. XRY is a tangent to the circle at point R. OZ is produced to meet at Y where OY ||QR. $R\hat{O}Y = 20°$ and $S\hat{R}O = 10°$. Chord SZ is drawn.



4.1 Calculate, with reasons, the size of the following angles:

4.1.1 $\hat{S_1}$

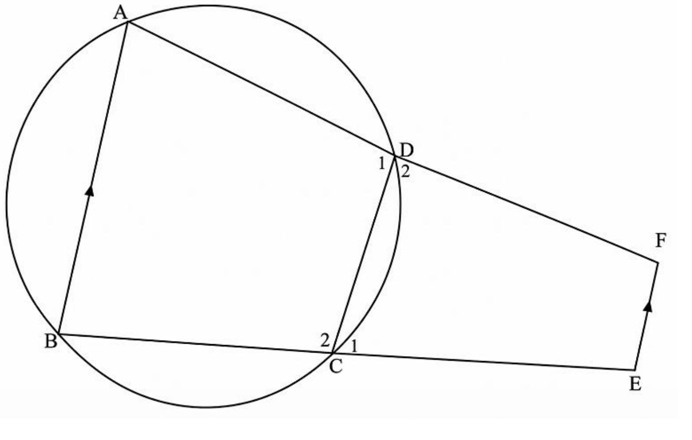
4.1.2 $\widehat{R_3}$

4.1.2 $\hat{P}$

4.1.2 $\hat{S_2}$

4.2 Prove that XRY is a tangent to the circle passing through R, T and O.

5. In the diagram below, ABCD is a cyclic quadrilateral. Chords AD and BC are produced to F and E , respectively. F and E are joined such that EF || AB.



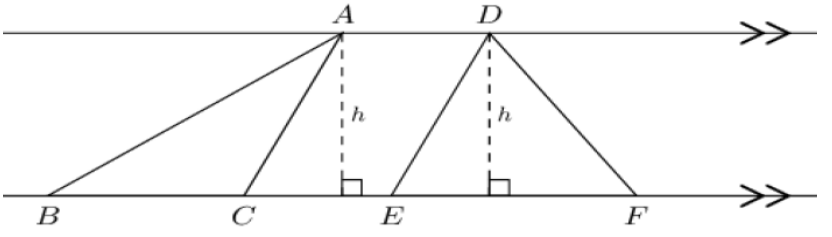
Prove that CEFD is a cyclic quadrilateral.

Similarity and Proportionality

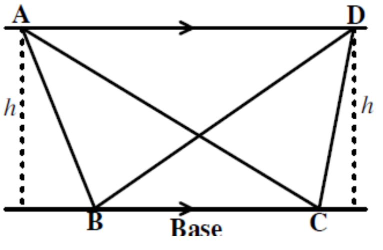
Proportionality

Area of Triangles

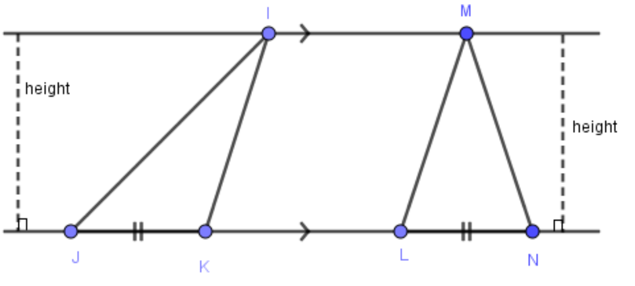
1. In the diagram below, △ABC and △DEF have the same height (h) since both triangles are between the same parallel lines.



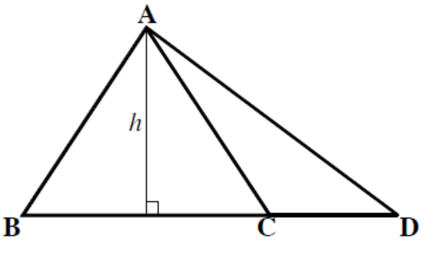
2. If triangles have the same base and same height, **then their areas are equal.**



3. Triangles with equal or common bases lying between parallel lines have the same area.



4. If triangles have the same height (common vertex), but their bases not equal, **then the ratio of their areas is equal to the ratio of their bases.**

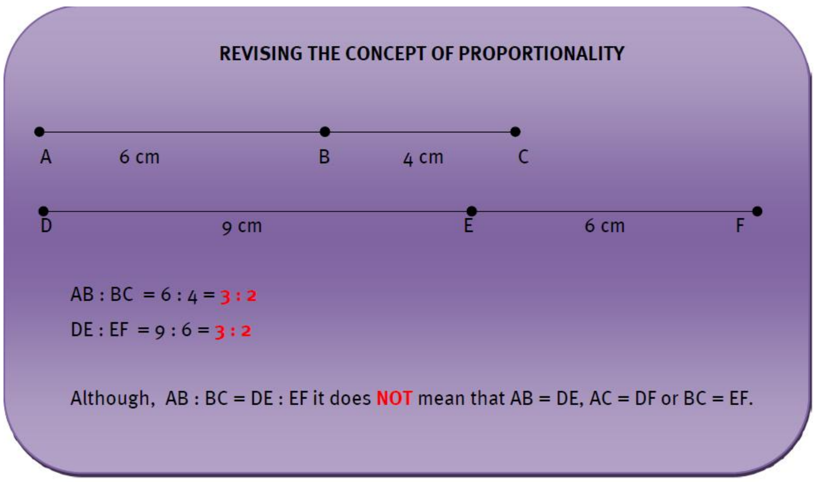


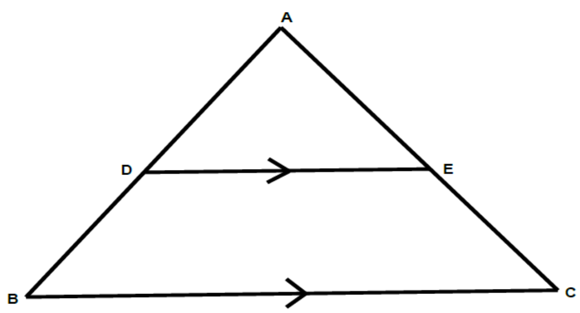
e.g Area of ΔABC : Area of $\Delta ABD = BC : BD$

Useful strategies in solving problems with proportion involving areas of triangles:

1.	If the two triangles in question have a common height and a common vertex.	Use area formula: Area $= \frac{1}{2}$base $\times$ height
2.	If the two triangles in question have a common angle.	Use area rule: Area $= \frac{1}{2}$a b sin C
3.	If none of the above.	Identify a common triangle and relate the two triangles in question to it, then use any of the two methods mentioned above.

A ratio describes the relationship between two quantities which have the same units. We can use ratios to compare distance, age, etc. A ratio is a comparison between two quantities of the same kind and has no units. If two or more ratios are equal to each other, then we say that they are in the same **proportion**.



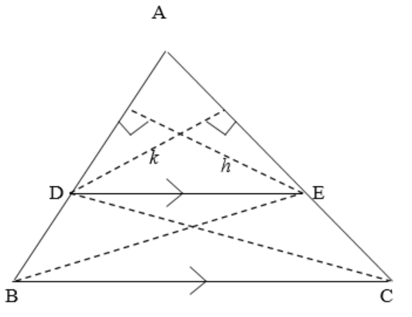
Theorem 1	**Converse**
A line drawn parallel to one side of a triangle divides the other two sides proportionally. **<u>(Line || one side of Δ OR Prop theorem; name || lines)</u>**  If DE || BC, then $\frac{AD}{BD} = \frac{AE}{EC}$ Other corollaries from this theorem: (1) $\frac{AB}{AD} = \frac{AC}{AE}$ (2) $\frac{BD}{DA} = \frac{CE}{EA}$ (3) $\frac{AB}{DB} = \frac{AC}{EC}$ (4) $\frac{BD}{BA} = \frac{CE}{CA}$	If a line cuts two sides of a triangle proportionally, then that line is parallel to the third side. **<u>(Line divides sides of Δ proportionally.)</u>** If $\frac{AD}{BD} = \frac{AE}{EC}$, then $DE \|\| BC$.

NB: The midpoint theorem, which you learned about in grade 10, is a special case of this theorem.

The proof of Theorem 1 is examinable but not the proof of the converse.

A line drawn parallel to one side of a triangle divides the other two sides proportionally.

(line // one side of *Δ*)



Given: ΔABC, D on AB and E on AC with

DE // BC. Draw DC and BE.

R.T.P: $\frac{AD}{DB} = \frac{AE}{EC}$

Construction: In ΔADE, draw altitudes h and k to bases AD and AE respectively.

Proof:

$$\frac{Area\ \Delta ADE}{Area\ \Delta BDE} = \frac{\frac{1}{2}.AD.h}{\frac{1}{2}.DB.h} \qquad \frac{Area\ \Delta ADE}{Area\ \Delta CED} = \frac{\frac{1}{2}.AE.k}{\frac{1}{2}.EC.k}$$

$$= \frac{AD}{DB} \qquad = \frac{AE}{EC}$$

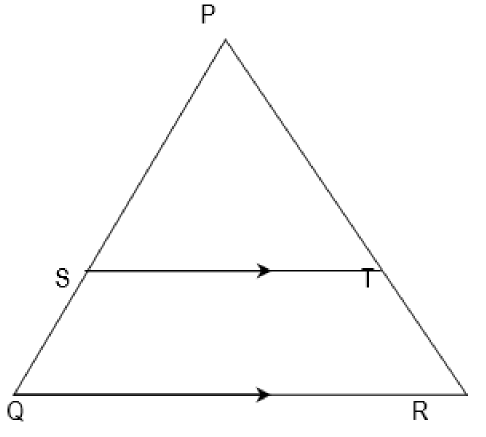
but Area ΔBDE = Area ΔCED (same base, same height)

$$\therefore \frac{Area\ \Delta ADE}{Area\ \Delta BDE} = \frac{Area\ \Delta ADE}{Area\ \Delta CED}$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}$$

Example 1

In ΔPQR, ST // QR.



Calculate:	
1.1	PT if PQ = 28 mm, PR = 21 mm and PS = 16 mm
1.2	PT and TR if PR = 18 mm, SQ = 10 mm and PS = 12,5 mm

1.1 $\frac{PS}{QP} = \frac{PT}{PR}$ [Line || one side of Δ]

$\frac{16}{28} = \frac{PT}{21}$

$PT = \frac{16}{28} \times 21$

$\text{PT} = 12 \text{ mm}$

1.2 $\frac{TR}{PR} = \frac{SQ}{PQ}$ [Line || one side of Δ]

$\frac{\text{TR}}{18} = \frac{10}{22.5}$

$TR = \frac{10}{22.5} \times 18$

$\text{TR} = 8 \text{ mm}$

$\frac{PT}{PR} = \frac{PS}{PQ}$ [Line || one side of Δ]

$\frac{\text{TR}}{18} = \frac{12.5}{22.5}$

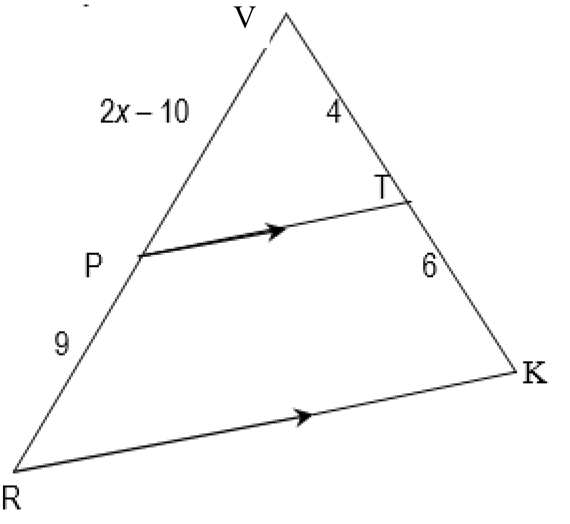
$x = \frac{12.5}{22.5} \times 18$

$= 10 \text{ mm}$

Example 2

In the diagram below, ΔVRK has P on VR and T on VK such that PT || RK. VT = 4 units,

PR = 9 units, TK = 6 units and VP = 2x – 10 units



Calculate the value of x.

Solution

$\frac{PV}{PR} = \frac{VT}{TK}$ [Line || one side of Δ]

$\frac{2x-10}{9} = \frac{4}{6}$

$2x - 10 = \frac{4}{6} \times 9$

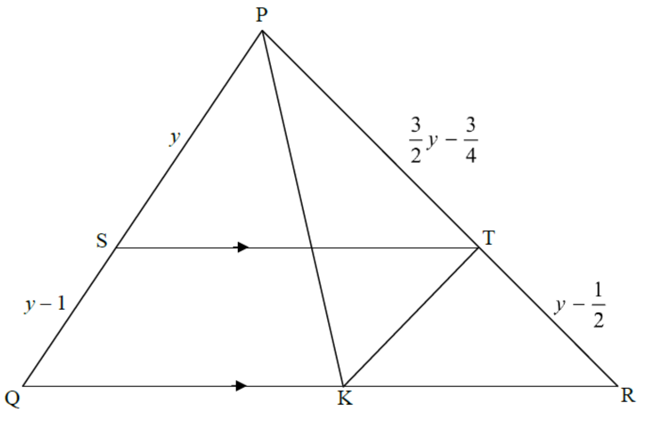
$2x - 10 = 6$

$2x = 16$

$x = 8\, units$

Example 3

In the diagram is ΔPQR with ST||QR. K is a point on QR and is joined to T and P. $PS = y, SQ = y - 1, TR = y - \frac{1}{2}$ and $PT = \frac{3}{2}y - \frac{3}{4}$ units



3.1.Calculate, with reasons, the value of y.
3.2.If $P\widehat{K}T = Q$, prove that PSKT is a cyclic quadrilateral.

Solution

3.1

$$\frac{y}{y-1} = \frac{\frac{3}{2}y - \frac{3}{4}}{y - \frac{1}{2}} \quad [\text{prop.th ; ST} \parallel \text{QR}]$$

$$y\left(y - \frac{1}{2}\right) = (y-1)\left(\frac{3}{2}y - \frac{3}{4}\right)$$

$$y^2 - \frac{1}{2}y = \frac{3}{2}y^2 - \frac{9}{4}y + \frac{3}{4}$$

$$4y^2 - 2y = 6y^2 - 9y + 3$$

$$0 = 2y^2 - 7y + 3$$

$$0 = (y-3)(2y-1)$$

$$y = 3 \text{ or / } of \;\; y \neq \frac{1}{2}$$

3.2

$$\hat{PST} = \hat{Q} \quad [\text{corresp} \angle\text{s ; ST} \parallel \text{QR}]$$

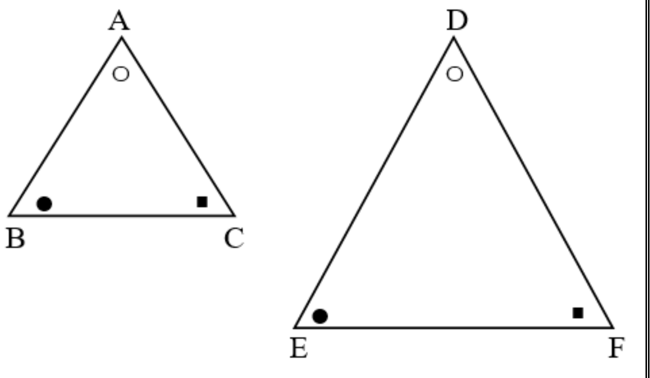
$$\hat{PST} = \hat{PKT} = \hat{Q}$$

P, S, K and T are consyclic

[converse $\angle$s in same segment]

$\therefore$ PSKT is a cyclic quadrilateral

Similarity of Triangles

Two triangles are similar if: 1. all three pairs of corresponding angles are equal $\hat{A} = \hat{D}$, $\hat{B} = \hat{E}$, $\hat{C} = \hat{F}$ 2. all pairs of corresponding sides are in the same proportion. $\frac{AB}{DF} = \frac{AC}{DF} = \frac{BC}{EF}$	 $\Delta ABC \;/\!/\!/\; \Delta DEF$ $\rightarrow$ (AAA or $\angle\angle\angle$ meaning Equiangular triangles)

To prove two triangles similar, you only need to prove one of the following:

1. Prove **“three”** equal angles (If any angles are given)

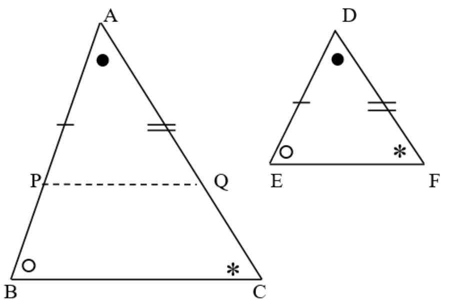
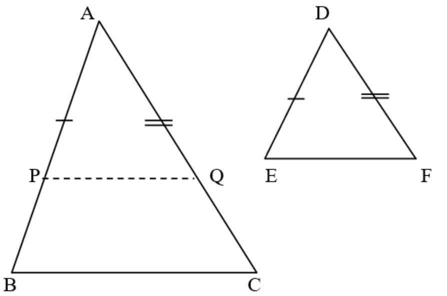
- If two pairs of angles are equal, the third pair must be equal because of the sum of angles in a triangle. You only need to find two equal angles

2. Prove that the sides are in proportional **(If any side lengths are given)**,

- You need to prove that all **THREE** ratios are equal.

Once the triangles are similar from AAA, write the sides that are in proportion.

Always write letters for corresponding vertices in a corresponding pattern when naming similar triangles.

Theorem 2	Converse
If two triangles are **equiangular** then their corresponding sides are **in proportion** and the triangles are **similar**. <u>**(||| Δs OR equiangular Δs)**</u>  If in ΔABC and ΔDEF, $\hat{A} = \hat{D}$, $\hat{B} = \hat{E}$ and $\hat{C} = \hat{F}$ Then $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$	If the corresponding sides of two triangles are in the same proportion, the two triangles are equiangular and therefore similar. <u>**(Sides of Δ in prop)**</u>  If in ΔABC and ΔDEF, $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$ Then $\Delta ABC \,\|\|\|\, \Delta DEF$

Ways in which similarity can be asked

1. Prove that ΔABC|||ΔDEF.

2. Prove that $\frac{AB}{DE} = \frac{BC}{EF}$. First prove: ΔABC|||ΔDEF and then deduce the proportion of the sides.

3. Prove that: KN. PX = NR. YP. Find two triangles in which KN, PX, NR and YP

(or sides equal to these) and thus prove that: ΔKNR|||ΔYPX, then deduce what you were asked to prove.

Strategies in locating the triangles when proving that they similar

1. Try taking letters from the top and bottom and see if you can locate similar triangles

$$\frac{BC}{AC} = \frac{AB}{AD}$$

   Use the diagram to check whether ABC and ABD form triangles, if yes, prove equal angles. If not, you go for option 2.

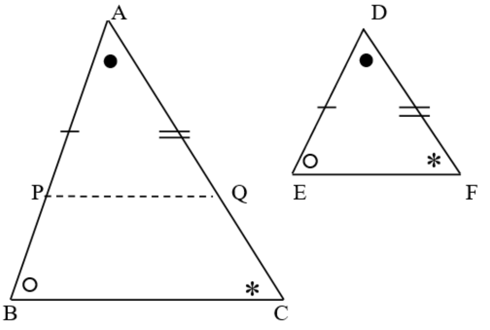
2. Try taking letters from the left and right and see if you can locate similar triangles

$$\frac{BC}{AC} = \frac{AB}{AD}$$

   Use the diagram to check whether ABC and ACD form triangles, If yes, prove equal angles. If not, you go for option 3.
3. If you cannot locate triangles, try one or more of the following strategies:
   a) Replace lengths with equal other lengths and then try to locate triangles.
   b) Look for other pairs of triangles which might be similar and have a bearing on what you are trying to prove.

The proof of similarity is examinable but not the proof of its converse.

If two triangles are equiangular, then the triangles are similar. (equiangular Δ s)



Given: ΔABC and ΔDEF with $\hat{A} = \widehat{D}$, $\hat{B} = \hat{E}$ and $\hat{C} = \hat{F}$

R.T.P: $\Delta ABC|||\Delta DEF$

Construct PQ such that AP = DE and AQ = DF

Proof: In ΔAPQ and ΔDEF

AP = DE (construction)

$\hat{A} = \hat{D}$ (given)

AQ = DF (construction)

$\therefore \Delta APQ \equiv \Delta DEF$ (SAS)

$\therefore \hat{P}_1 = \hat{E}$

But $\hat{B} = \hat{E}$ (given)

$\therefore \hat{P}_1 = \hat{B}$

$\therefore$ PQ // BC |Corr. $\angle$'s equal

$\therefore \frac{AP}{AB} = \frac{AQ}{AC}$

But AP = DE and AQ = DF

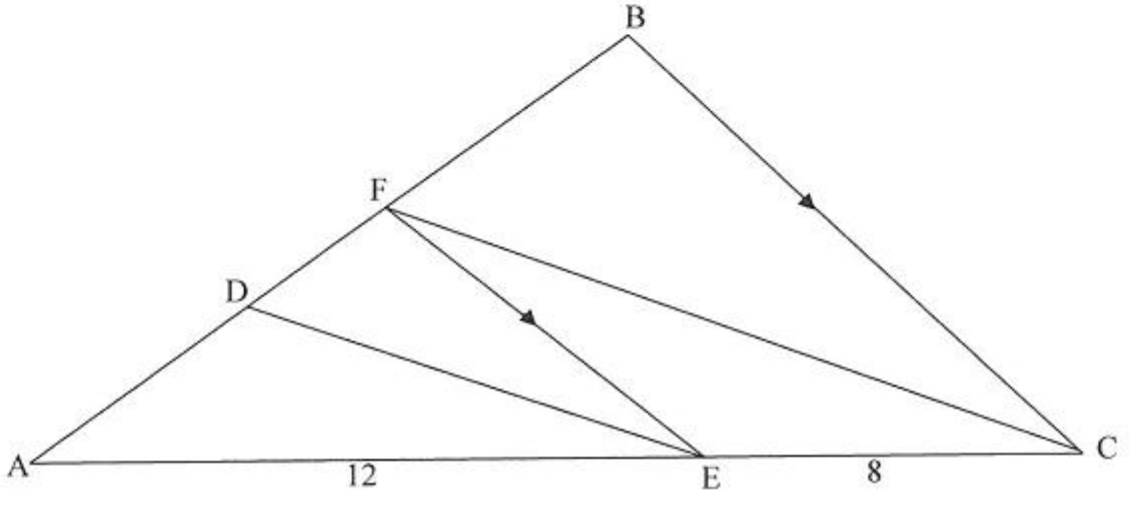
$\therefore \frac{DE}{AB} = \frac{BF}{AC}$

Example 1

1 Complete the following statement of the theorem in the ANSWER BOOK:

If a line divides two sides of a triangle in the same proportion, then ...

2 In the diagram ABC is a triangle with F on AB and E on AC. BC | | FE. D is on AF with $\frac{AD}{AF}=\frac{3}{5}$. AE = 12 units and EC = 8 units.



2.1 Prove that DE | | FC.

2.2 If AB = 14 units, calculate the length of BF.

Solution:

1. the line is parallel to the third side of the triangle.

2.1 $\frac{AE}{AC}=\frac{12}{20}=\frac{3}{5}$

$\frac{AD}{AF}=\frac{3}{5}$

$\therefore \frac{AE}{AC}=\frac{AD}{AF}$

$\therefore DE \parallel FC$ (line divides two sides of Δ in prop)

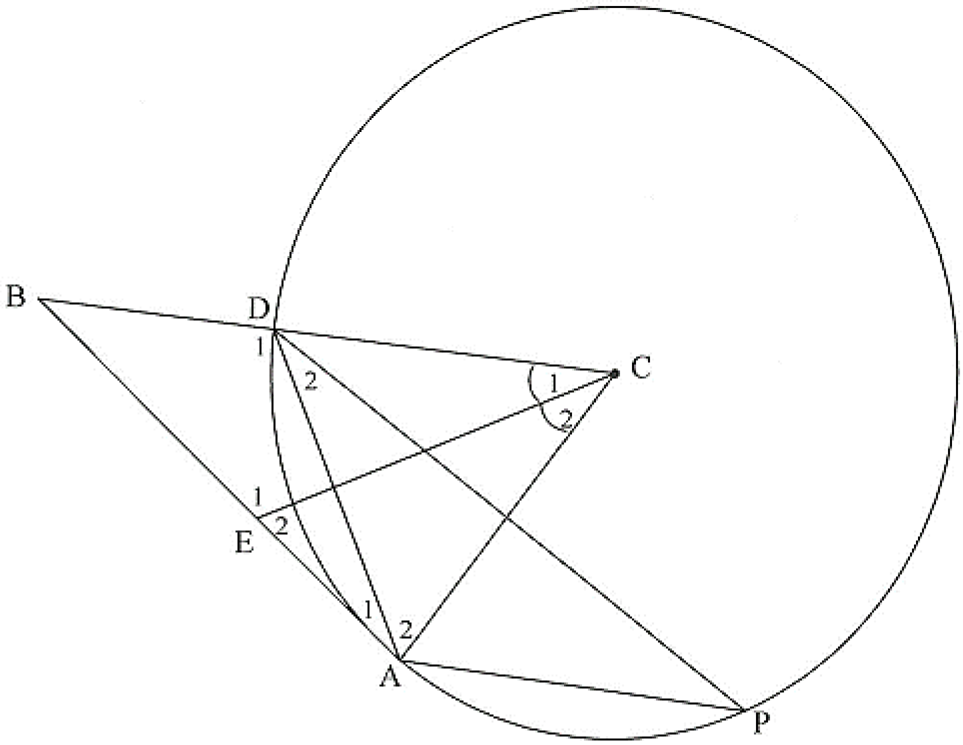
2.2 $\frac{BF}{BA}=\frac{8}{20}$ (prop theorem $BC \parallel FE$)

$\therefore BF=\frac{8}{20}(14)$

$\therefore BF=\frac{28}{5}$ **OR** $FB=5\frac{3}{5}$ **OR** $FB=5{,}6$

Example 2

In the diagram C is the centre of the circle DAP. BA is a tangent to the circle at A. CD is produced to meet the tangent to the circle at B. DP and DA are drawn. E is a point on BA such that EC bisects $D\hat{C}A$. Let $\hat{C}_1 = x$.



1 Prove that $\Delta BAD \ ||| \ \Delta BCE$.

2 If it is also given that $AB = 8$ units and $AC = 6$ units, calculate:

(a) The length of BD

(b) The length of BE

(c) The size of x

Solution

1 $D\hat{C}A = 2x$ (EC bisector)

$\hat{P} = x$ ($\angle$ at centre = 2 $\times\angle$ at circumference)

$\hat{A}_1 = \hat{P} = x$ (tangent-chord theorem)

In ΔBAD and ΔBCE:

$\hat{B} = \hat{B}$ (common)

$\hat{A}_1 = \hat{C}_1$ (proven above)

$\therefore \Delta BAD \mid\mid\mid \Delta BCE$ ($\angle\angle\angle$)

2(a) $B\hat{A}C = 90°$ (tangent $\perp$ radius)

$\therefore BC^2 = 8^2 + 6^2 = 100$ (Pythagoras theorem/)

$BC = 10$

$AC = DC = 6$ (radii)

$\therefore$ $BD = 10 - 6 = 4$ units

2(b) $\frac{BA}{BC} = \frac{BD}{BE}$ ($\Delta BAD \mid\mid\mid \Delta BCE$)

$\therefore \frac{8}{10} = \frac{4}{BE}$

$\therefore$ $BE = 5$ units

2(c) $AE = 3$

In ΔACE:

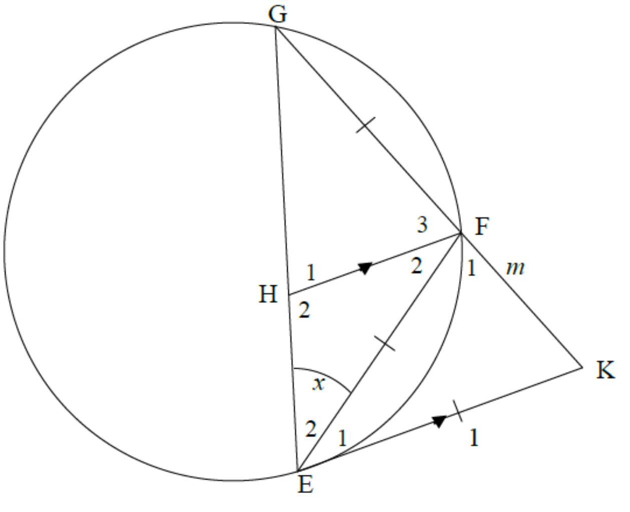
$\tan x = \frac{3}{6}$

$\therefore x = 26{,}57°$

Exercise 3

Question 1

In the diagram, E, F and G are points on the circle. KE is a tangent to the circle at E. KFG is a straight line. H is a point on chord GE such that HF || EK. KE = EF = FG. KF = m units and KE = 1 unit. $\hat{E}_2 = x$.



1.1 Name, giving reasons, THREE other angles each equal to x.

1.2 Express $\widehat{K}$ in terms of x.

1.3 Calculate, with reasons, the size of x.

1.4 Prove that:

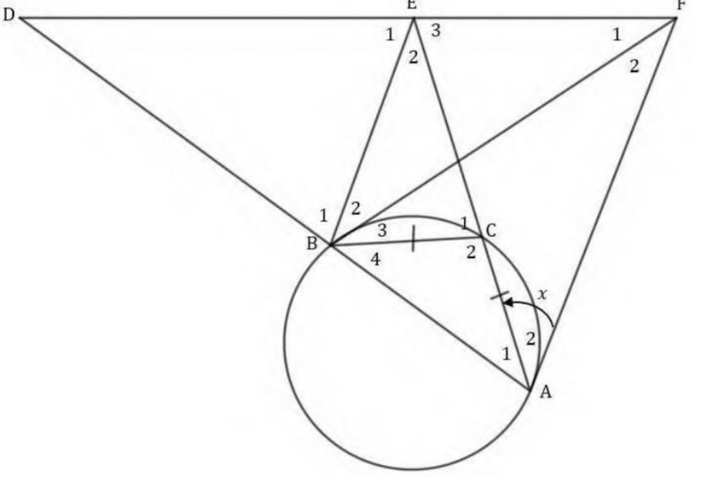
1.4.1 ΔKGE ||| ΔKEF

1.4.2 $m^2 + m - 1 = 0$

1.5 If it is given that GH = 2 units, calculate the length of HE, rounded off to 2 decimal places.

Question 2

FA and FB are tangents to the circle ABC with $BC = AC$. $FD \parallel CB$ and $C\hat{A}F = x$. Chord AB is produced to D and chord AC is produced to meet DF at E. BC is joined.



2.1 Write down FIVE other angles equal to x.

2.2 Hence deduce that:

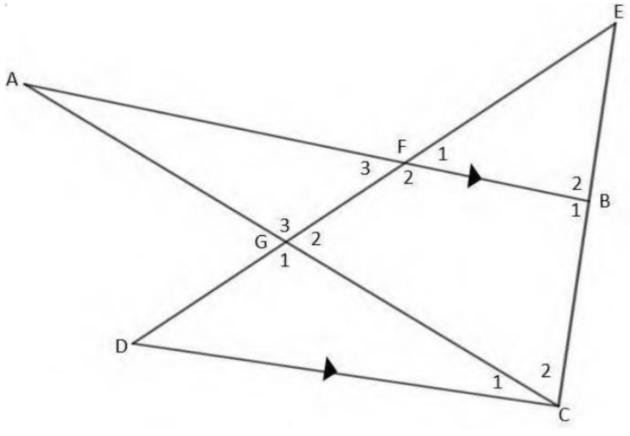
2.2.1 ABEF is a cyclic quadrilateral.

2.2.2 $AF = BD$

2.3 Prove that $AF: FB = DE: AE$

Question 3

In the diagram below, F lies on lines ED and AB and G lies on lines ED and AC. DGFE is a straight line and: BFA∥DC, AB = 40 cm , BC = 20 cm , EF = 16 cm , EB = 10 cm and FB = 12 cm



3.1.With reasons, determine the value of $\frac{EF}{ED}$

3.2.Determine the length of ED.

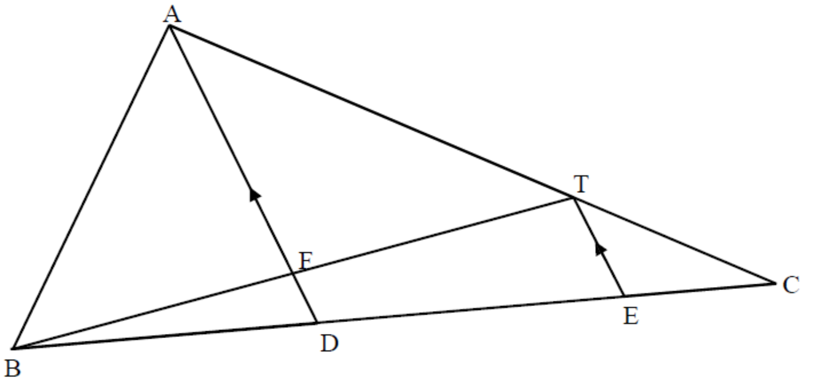
3.3.Without any reasons complete: Δ EFB||| Δ $\cdots$

3.4.Hence, with reasons, determine the length of DC.

3.5.Show that $\frac{3\ Area\ of\ \Delta EFB}{16\ Area\ of\ \Delta DGC} = \frac{1}{DG}$

Question 4

In the figure, D is a point on side BC of □ABC such that BD = 6 cm and DC = 9 cm. T and E are points on AC and DC respectively such that TE||AD and AT : TC = 2 : 1



4.1 Show that D is the midpoint of BE.

4.2 If FD = 2 cm, calculate the length of TE.

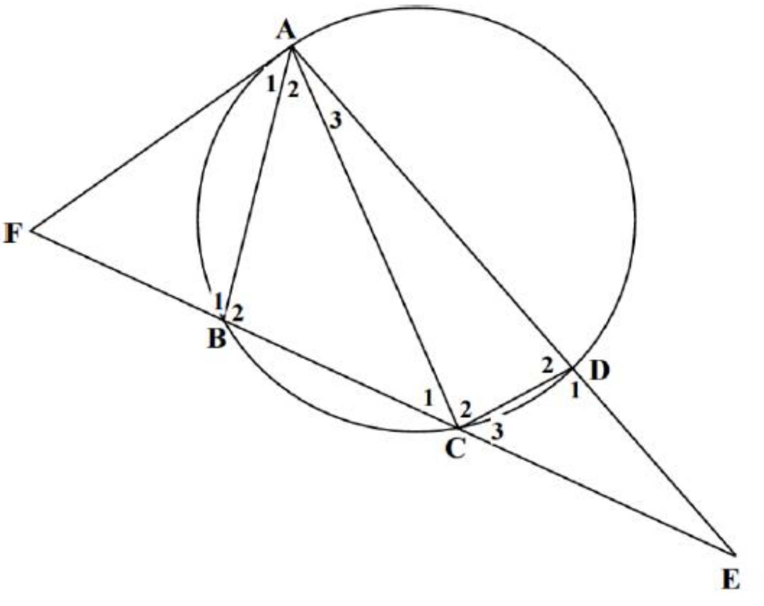
4.3 Calculate the numerical value of:

4.3.1 $\frac{Area\ of\ \Delta ADC}{Area\ of\ \Delta ABC}$

4.3.2 $\frac{Area\ of\ \Delta TEC}{Area\ of\ \Delta ABC}$

Question 5

FA ia a tangent to circle ABCD at A. AD and FBC produced meet at E.



5.1 Prove that ΔDEC|||ΔBEA.

5.2 Prove that ΔFAB|||ΔFCA.

5.3 Hence show that FA.CA= FC.AB.

BIBLIOGRAPHY

1	Study guide Via Afrika mathematics Grade 12
2	CURRICULUM AND ASSESSMENT POLICY STATEMENT(CAPS) SENIOR PHASE GRADES (7 – 9) MATHEMATICS
3	CURRICULUM AND ASSESSMENT POLICY STATEMENT(CAPS) FET PHASE GRADES (10 – 12) MATHEMATICS
4	MATHEMATICS QUESTION PAPER GRADE 9 2013 EASTERN CAPE
5	MATHEMATICS QUESTION PAPER GRADE 9 2014 EASTERN CAPE
6	MATHEMATICS QUESTION PAPER GRADE 9 2016 EASTERN CAPE
7	MATHEMATICS QUESTION PAPER GRADE 9 2011 EASTERN CAPE
8	BISHOPS DIOCESAN COLLEGE GRADE 9 PAPER 2 2017
9	2021 Grade 12 MATHEMATICS EXAMINATION GUIDELINES
10	MATHEMATICS QUESTION PAPERS GRADE 12 (2008 – 2017) NATIONAL
11	MIND ACTION SERIES TEXT BOOK GRADE 12
12	MATHEMATICS QUESTION PAPER GRADE 9 2017 - 2020 GAUTENG
13	WWW.MATHISFUN.COM