Training and Consultancy

The path to enlightened education

SUBJECT: TECHNICAL MATHEMATICS

GRADE 12

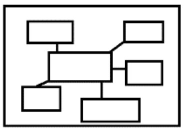
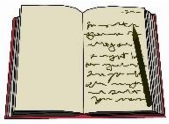
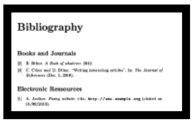
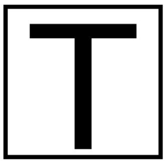
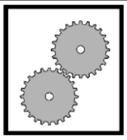
AUTUM CLASSES

TEACHER AND LEARNER CONTENT MANUAL

DIFFERENTIAL CALCULUS

A Division of Marematlou Group Holdings

ICON DESCRIPTION

 MIND MAP	 EXAMINATION GUIDELINE	 CONTENTS	 ACTIVITIES
 BIBLIOGRAPHY	 TERMINOLOGY	 WORKED EXAMPLES	 STEPS

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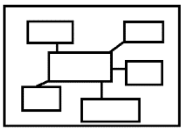
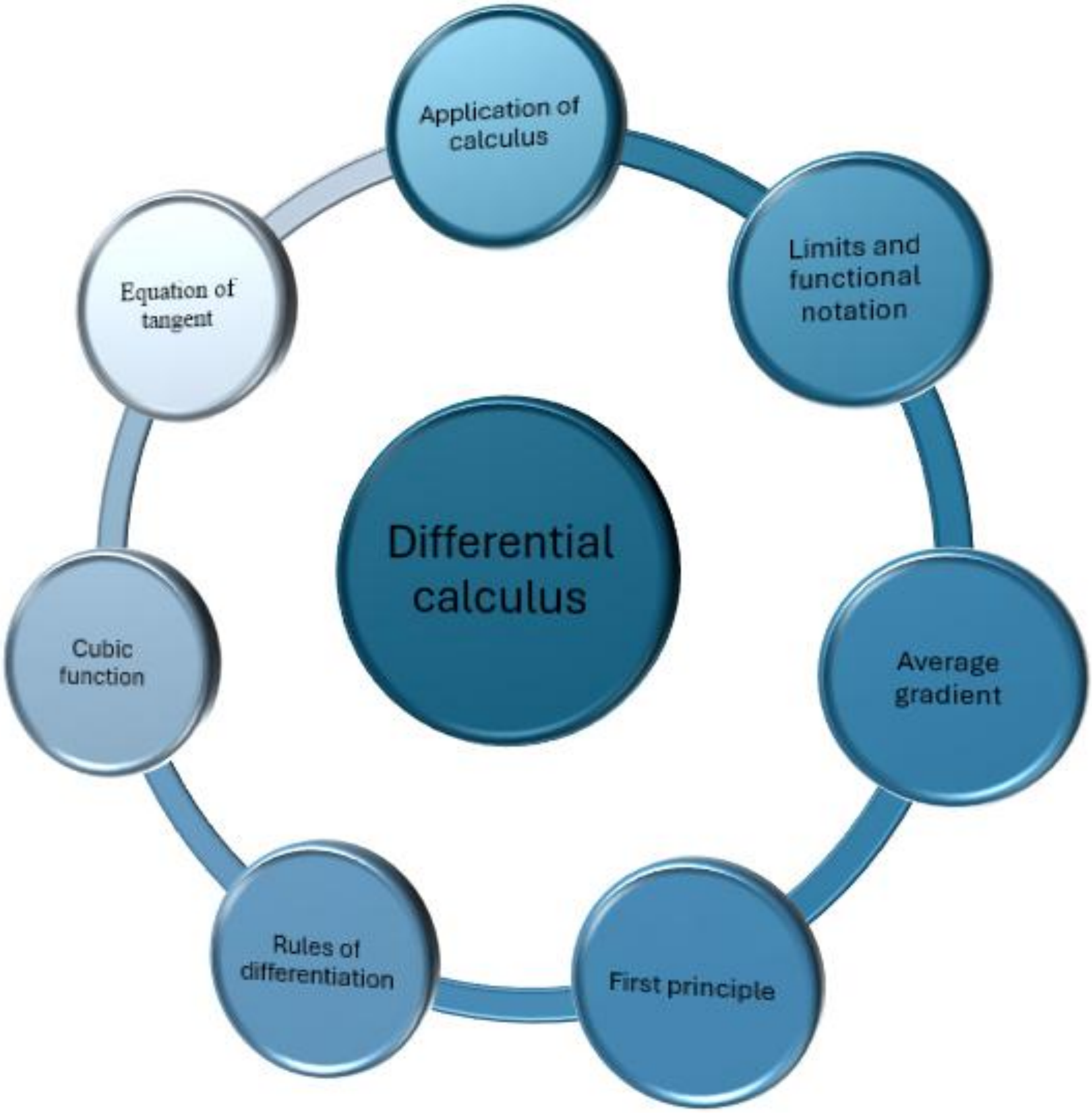



Application of calculus
Limits and functional notation
Average gradient
First principle
Rules of differentiation
Cubic function
Equation of tangent
Differential calculus

1. LIMITS AND FUNTIONAL NOTATION

A limit is a value that a function gets closer to when the input gets closer to a certain value. When finding a limit, we look at what happens to a function value of a curve (the y – value) as we get closer and closer to a specific x – value on the curve.

When writing a limit, we use the notation $\lim_{x \to c} f(x)$, where 'lim' indicates that we are determining the limit, the notation $x \to c$ informs us which specific value the x – value is approaching and $f(x)$ represents the function with which we are working.

Substitute c into $f(x)$, taking into consideration that the function does not become undefined. In the case of UNDEFINED, simplify $f(x)$ first then substitute c.

$$\lim_{x \to 0} \frac{2x^2 + 4x}{x}$$

$$\lim_{x \to 0} \frac{2x^2}{x} + \frac{4x}{x}$$

$$\lim_{x \to 0} 2x + 4$$

$$= 2(0) + 4$$
$$= 4$$

Functional notation:

- Remember that $f(x)$ is read as f of x.
- By substituting values of x into the function $f(x)$, one can obtain a set of coordinates $(x; y)$ or $(x; f(x))$

Given $f(x) = 2x + 1$, calculate:

a) $f(4)$

b) $f(x+1)$

Solutions:

a) $f(4) = 2(4) + 1$
$= 9$

b) $f(x+1) = 2(x+1) + 1$
$= 2x + 2 + 1$
$= 2x + 3$

ACTIVITY 1

1. Simplify:

   1.1. $\lim\limits_{x \to 4} 1 - 3x$

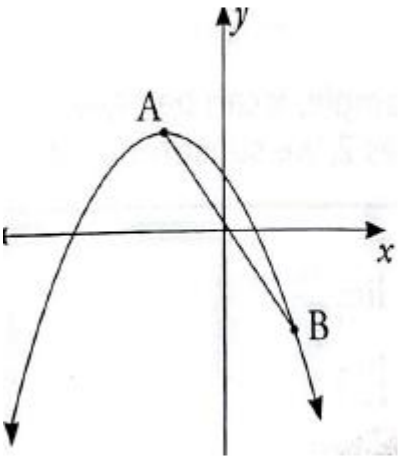
   1.2. $\lim\limits_{x \to 1} \dfrac{x^2 + 3x - 4}{x - 1}$

2. Given $f(x) = x^2 - 4$, determine $f(x+1)$.

2. AVERAGE GRADIENT

The gradient of a linear function is simply determined by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ but when we work with curves, the gradient changes at each point along the curve. So, we need to determine the average gradient.

Average gradient between two point on a curve is the gradient of the straight line passing through those two points.



Therefore, the average gradient will be: $m_{ave} = \dfrac{f(b) - f(a)}{b - a}$

Find the average gradient of $f(x)=x^2-16$ between $x=5$ and $x=9$.

Solution:

$$m_{ave}=\frac{f(b)-f(a)}{b-a}$$
$$=\frac{(9)^2-16-\left((5)^2-16\right)}{9-5}$$
$$=\frac{65-9}{4}$$
$$=14$$

ACTIVITY 2

1. Determine the average gradient of $g(x)=x^3-7x^2-10x+16$ between $x=-2 \quad and \quad x=4$.

2. A ball thrown out the window from the third floor of a building has a height of $h(t)=t^2-9t+20$ meters after t seconds. Calculate the average velocity of the ball between $2\sec \quad and \quad 4\sec$.

3. FIRST PRINCIPLES

From the graph above, line AB is a secant. If point B is moved towards point A, then the distance (h) between these two points becomes smaller and smaller thus changing line AB from being a secant to being a tangent.

When the latter happens and the two points now share the same position, the we no longer talk about the average gradient. We now talk about the instantaneous gradient (gradient at a point), which is also the gradient of the tangent or the DERIVATIVE of a function (the limit of the average gradient).

The derivative can be written as $f'(x)$ and is defined by:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The process of finding the derivative is called differentiation. When we use the definition above to differentiate a function, it is said that we differentiating using FIRST PRINCIPLES.

- Copy the definition (formula) EXACTLY AS IT IS from the formula sheet.
- Substitute $f(x)$ and $f(x+h)$. REMEMBER WHAT YOU LEARNT UNDER FUNCTIONAL NOTATION.
- Simplify by breaking the brackets.
- Simplify by grouping like terms
- Simplify the fraction and write the final answer

Determine the derivative of $f(x) = 1 - x$ using FIRST PRINCIPLES.

Solution:

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{1-(x+h)-(1-x)}{h} \\
&= \lim_{h \to 0} \frac{1-x-h-1+x}{h} \\
&= \lim_{h \to 0} \frac{-h}{h} \\
&= \lim_{h \to 0} -1 \\
&= -1
\end{aligned}
$$

ACTIVITY 3

1. Determine using FIRST PRINCIPLES the derivative of the following
   1.1. $f(x) = 2 - 3x$
   1.2. $f(x) = 7x + 1$
   1.3. $f(x) = 3$

4. POWER RULE

We will now determine the derivative using the power rule. The following notations are used for the derivative:

$\frac{dy}{dx}$ if $y = \ldots$ is given

$f'(x)$ if $f(x)$ is given

$D_x[\ldots]$

All the above notations say we need to differentiate with respect to x.

Necessary prior knowledge:

- How to multiply brackets
- How change surd form to exponential form: $\sqrt[n]{x^m} = x^{\frac{m}{n}}$
- How to remove a variable from being a denominator: $\frac{1}{a^n} = a^{-n}$

NOTE: you may not differentiate using the power rule while your expression has brackets, surds and/or your variable is a denominator.

If $f(x) = ax^n$, then the derivative will be $f'(x) = anx^{n-1}$ (Power rule)

Simplify the given expression by removing brackets, changing surds and making sure that your variable is not a denominator, then:

Multiply the coefficient by the exponent, then subtract 1 from the exponent

Determine: $D_x\left[\dfrac{2\left(\sqrt{x}+1\right)}{x}\right]$

Solution:

$$D_x\left[\frac{2\left(x^{\frac{1}{2}}+1\right)}{x}\right]$$

$$D_x\left[\frac{2x^{\frac{1}{2}}+2}{x}\right]$$

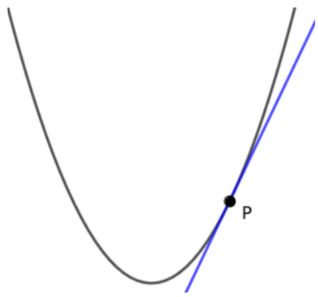
$$D_x\left[2x^{\frac{1}{2}}+2x^{-1}\right]$$

$$= x^{-\frac{1}{2}} - 2x^{-1}$$

ACTIVITY 4

1. Determine $D_y\left[(y+2)^2\right]$
2. Given $xy = \sqrt{x^3} - 2$:
   - 2.1. Simplify $xy = \sqrt{x^3} - 2$
   - 2.2. Hence, determine $\dfrac{dy}{dx}$

5. EQUATION OF THE TANGENT



- P is a point of tangency (point of contact).
- A tangent is a straight line; therefore, we determine its equation easily by using: $y - y_1 = m(x - x_1)$
- m is the instantaneous gradient (derivative)
- $(x_1 ; y_1)$ point of tangency

How to determine the equation of a tangent.

Given the function $f(x)$ and the x – coordinate of the point of tangency. Do the following 3 substitutions:

1. Substitute the given x value into the derivative $f'(x)$ to find the gradient m.
2. Substitute the given x value into $f(x)$ to obtain the y – coordinate of the point of tangency $(x_1 ; y_1)$.
3. Substitute the obtained m and $(x_1 ; y_1)$ into $y - y_1 = m(x - x_1)$ and simplify.

Determine the equation of the tangent to the curve $g(x) = 5x^3 + x^2$ at $x = 1$.

Solution:

$g'(x) = 15x^2 + 2x$

$g'(1) = 15(1)^2 + 2(1)$

$\quad = 17$

$g(1) = 5(1)^3 + (1)^2$

$\quad = 6 \qquad (1 ; 6)$

$y - 6 = 17(x - 1)$

$y = 17x - 17 + 6$

$y = 17x - 11$

How to determine the coordinates of the point of tangency (point of contact).

Given the function $f(x)$ and the gradient. Do the following 2 substitutions:

1. Substitute the given gradient m into the derivative $f'(x)$ to find the x coordinate.
2. Substitute the obtained x coordinate into $f(x)$ to obtain the y – coordinate of the point of tangency $(x_1; y_1)$.

Determine the point of contact between the tangent with the gradient of 10 and the curve $f(x) = x^2 + 2x - 1$.

Solution:

$f'(x) = 2x + 2$

$10 = 2x + 2$

$8 = 2x$

$4 = x$

$f(4) = (4)^2 + 2(4) - 1$

$\quad = 23$

$\therefore$ The point of contact is $(4; 23)$

ACTIVITY 5

1. Determine the equation of the tangent to curve of $f(x) = \frac{1}{x} + 1$ at $x = 1$.
2. The tangent with the gradient of 2 touches the curve of $g(x) = x^2 - 8x + 16$. Determine the coordinates of the point of tangency.

6. CUBIC FUNCTION

Sketching the cubic function:

To sketch the cubic function, determine these key features:

- Intercepts (both x and y intercepts)
- Turning points (stationary points)

Sketch the graph, clearly indicating the intercepts with the axis as well as the turning points.

- y intercept
  Let $x=0$ and simplify.

- x intercept
  Let $y=0$ and solve a third-degree polynomial by inspection, long division or synthetic method. The latter will require you to use FACTOR THEOREM to find the divisor (factor).
  NOTE: there can be one, two or three x intercepts depending on the given function.

- Stationary / turning points of a function $f(x)$
  - Determine the derivative $f'(x)$
  - Equate the derivative to zero, $f'(x)=0$
  - Solve for x, then substitute the obtained values of x into the original function $f(x)$ to obtain the corresponding y values.

- Plot the points and join them

Given the function $f(x)=-x^3+5x^2+8x-12$:

1. Determine the y intercept of f.
2. Show that $x-1$ is a factor of f.
3. Hence, determine the $x-\text{int}$ of f.
4. Determine the coordinates of the turning points.
5. Sketch the graph of f, clearly indicate the intercepts with the axis as well as the turning points.

Solution:

1. $f(0)=-(0)^3+5(0)^2+8(0)-12$
   $=-12$

2. $x-1=0$
   $x=1$
   $f(1)=-(1)^3+5(1)^2+8(1)-12$
   $=0$
   $\therefore\ x-1$ is a factor.

3. $$\begin{array}{r|rrrr} 1 & -1 & 5 & 8 & -12 \\ & 0 & -1 & 4 & 12 \\ \hline & -1 & 4 & 12 & \lfloor 0 \end{array}$$
   $(x-1)(-x^2+4x+12)=0$
   $x=1$ or $x=\dfrac{-4\pm\sqrt{(4)^2-4(-1)(12)}}{2(-1)}$
   $x=1 \quad or \quad x=-2 \quad or \quad x=6$
   $(1;0) \quad (-2;0) \quad (6;0)$

4. $f'(x) = -3x^2 + 10x + 8$

$$-3x^2 + 10x + 8 = 0$$

$$x = \frac{-10 \pm \sqrt{(10)^2 - 4(-3)(8)}}{2(-3)}$$

$$x = -\frac{2}{3} \quad or \quad x = 4$$

$$\left(-\frac{2}{3} ; -\frac{400}{27}\right) \qquad (4 ; 36)$$

5.

ACTIVITY 6

Given the function $f(x) = x^3 - 6x^2 + 9x$:

1. Determine the y intercept of f.
2. Show that $x-3$ is a factor of f.
3. Hence, determine the $x-\text{int}$ of f.
4. Determine the coordinates of the turning points.
5. Sketch the graph of f, clearly indicate the intercepts with the axis as well as the turning points.

ACTIVITY 7

Given: $f(x) = -x^3 - x^3 + x + 10$.

1. Write down the coordinates of the y-intercept of f.
2. Show that (2 ; 0) is the only x-intercept of f.
3. Calculate the coordinates of the turning points of f.
4. Sketch the graph of f. Show all intercepts with the axes and all turning points.

7. APPLICATION OF CALCULUS

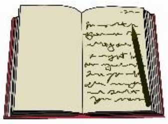
- Calculus can be used to solve optimisation and rate of change problems, this means we will have to find an optimum. The optimum value can be either a minimum or a maximum.
- The same principle used to determine stationary points will be used to determine the optimum value.

The displacement of a ball in meters from the centre of the playground after t seconds is given by $s(t) = 2t^2 - 25t + 150$

1. What is the initial position of the ball from the centre?
2. How fast was the ball moving at exactly 6 sec?
3. What is the minimum distance between the ball and the centre of the playground?

Solution:

1. $s(0) = 2(0)^2 - 25(0) + 150$
   $= 150$
2. $s'(t) = 4t - 25$
   $s'(6) = 4(6) - 25$
   $= -1\ m/s$
3. $0 = 4t - 25$
   $4t = 25$
   $t = \frac{25}{4}$
   $s\left(\frac{25}{4}\right) = 2\left(\frac{25}{4}\right)^2 - 25\left(\frac{25}{4}\right) + 150$
   $= 71{,}88\ m$



ACTIVITY 8

A tourist travels in a car over a mountainous pass during his trip. The height of the car above sea level, after t minutes, is given as $s(t) = 5t^3 - 65t^2 + 200t + 100$ meters. The journey lasts 8 minutes.

1. How high is the car above sea level when it starts its journey on the mountainous pass?
2. Calculate the car's average rate of change of height above sea level with respect to time in the first 2 minutes of the journey in the mountainous pass.
3. Calculate the velocity of the car 4 minutes after starting the journey on the mountainous pass.
4. After how many minutes will car reach its maximum height above sea level?
5. Hence, determine the maximum height that the car will reach.
6. How many minutes after the journey has started will the rate of change of height with respect to time be a minimum?

ACTIVITY 9

In a small factory the profit (P) per week is

$P = -x^3 + 108x + 1\,600$, where x is the number of workers. Calculate :

1. How many workers are necessary to assure a maximum profit?
2. The maximum profit

Additional activities

Average gradient, equation of a tangent and First Principle and differentiation by rules

ACTIVITY 10

1.1 Determine $f'(x)$, using FIRST PRINCIPLE if $f(x) = -x - 6$

1.2 Determine the following :

1.2.1 $f'(x)$ if $f(x) = \sqrt[3]{x^3} + 2x^{-2} - \frac{1}{2x}$

1.2.2 $\frac{d}{dx}\left[\frac{2x^2+3x-1}{x}\right]$

1.3 Determine the average gradient of $f(x) = x^2 - 1$ between the points, where $x = 2$ and $x = 6$.

1.4 Determine the equation of a tangent to the curve defined by $g(x) = 5x^2 - 3x$ at point where $x = 1$.

ACTIVITY 11

1.1 Determine the derivative of $f(x) = -\frac{1}{3}x - 4$ from FIRST PRINCIPLE.

1.2 Determine :

1.2.1 $D_x\left[x^5 + \frac{x^{-4}-2x}{2x^2}\right]$

1.2.2 $\frac{dy}{dx}$ as $y = ax^4 + \sqrt[3]{8x}$

1.3 Determine the value of k if the gradient of the tangent to a curve defined b, $f(x) = kx^2 - 4x + 5$ is equal to 16 at $x = -2$.

ACTIVITY 12

1.1 Determine $f'(x)$, using FIRST PRINCIPLE if $f(x) = -6k$.

1.2 Determine EACH of the following :

1.2.1 $\frac{dA}{dr}$ if $A = \pi r^2$

1.2.2 $D_x\left[\left(x - \sqrt{x}\right)^2\right]$

1.4 The equation of a tangent to the curve of the function g defined by $g(x) = ax^2 - x$ is $3x - y + 2 = 0$. The point of contact of the tangent and the curve of g is $(-1; -1)$. Determine the numerical value of a.

Sketching of cubic functions

ACTIVITY 13

Given : $f(x) = x^3 - 2x^2 - 4x + 8$

1.1 Show that $(x - 2)$ is a factor of the function $f(x)$.

1.2 Determine the $y -$ intercept of f.

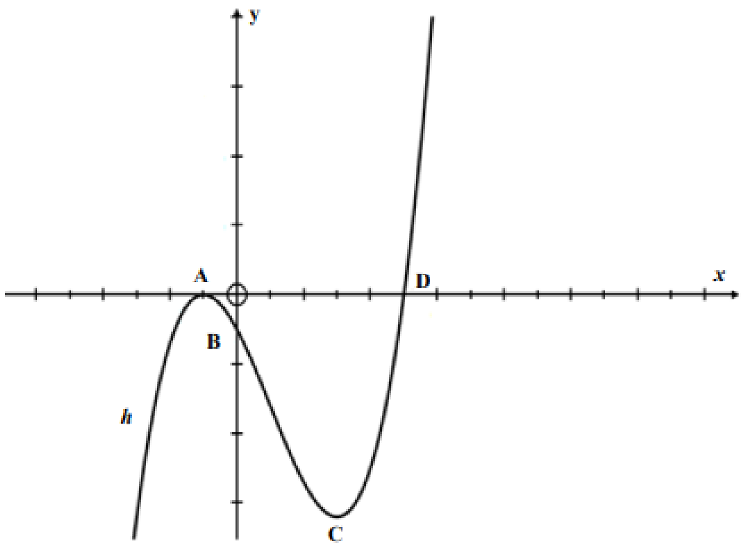
1.3 Determine the $x -$ intercept of f.

1.4 Determine the coordinates of the turning points of f.

1.5 Skech the graph of f , clearly show the turning points and the intercepts with the axes.

ACTIVITY 14

The graph below represents the function defined by $h(x) = x^3 - 3x^2 - 9x - 5$. A and C are the turning points of h. A and D cut the x –axis and the graph cuts the y –axis at point B.



1.1 Write down the coordinates of point B.

1.2 Show that $x + 1$ is a factor of h.

1.3 Hence, determine the coordinates of point D.

1.4 Determine the coordinates of the turning point C.

1.5 Write down the value(s) of x for which $h(x) > 0$.

ACTIVITY 15

Given: Function f defined by:
$f(x) = -(x-1)^2(x+3) = -x^3 - x^2 + 5x - 3$

1.1 Determine the $y -$ intercept of f.

1.2 Determine the $x -$ intercept of f.

1.3 Determine the coordinates of the turning points of f.

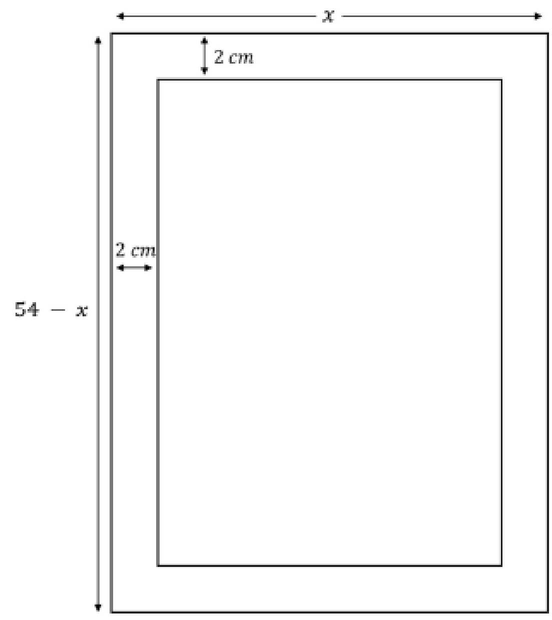
1.4 Skech the graph of f , clearly show the turning points and the intercepts with the axes.

1.5 Determine the values of x for which $f'(x) > 0$.

Application of calculus

ACTIVITY 16

The diagram below shows a sketch of a rectangular mirror inside a rectangular wooden frame. The wooden frame is 2 cm wide. The outer measurements of the frame are $(54 - x)$ cm in length and x cm in width.



1.1 Express the length and the breadth of the mirror in terms of x.

1.2 Show that the area of the mirror can be written as
$Area = -x^2 + 54x - 200$.

1.3 Calculate the maximum area of the mirror.

ACTIVITY 17

A printer produces bumper stickers for an election campaign. The cost to produce one sticker is $0{,}15 - 0{,}000002x$ rand, if x stickers are ordered (with $x < 10\ 000$) and the total cost to deliver the order is $0{,}095x - 0{,}0000005x^2$ rand.

1.1 Express the revenue in terms of x. Use the fact that $revenue = price\ per\ item \times number\ of\ items$ sold to express the revenue R(x) from x stickers as a product of two functions of x.

1.3 Determine the revenue when 5 000 stickers were produced

1.4 Determine the maximum revenue.

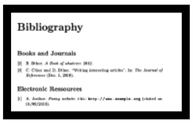
ACTIVITY 18

The displacement of a particle, in metres, from point B, after t seconds, is given by $s(t) = 2t^2 - 25t + 300$

1.1 How far is the particle from B initially?

1.2 How fast is the particle moving exactly 5 seconds?

1.3 What is the minimum distance between the particle and point B?


Bibliography
Books and Journals
Electronic Resources

BIBLIOGRAPHY

1	**NCS 2013 Feb/Mar Q10 paper 1**
2	**DBE/November 2020, Question 6**
3	**Technical mathematics Grade 12 self study guide Book 2**
4	