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Province of the
EASTERN CAPE
EDUCATION

MATHEMATICS

AUTUMN CLASSES

GRADE 12

TERM 1

TEACHER AND LEARNER CONTENT MANUAL

ICON DESCRIPTION

MIND MAP	EXAMINATION GUIDELINE	CONTENTS	ACTIVITIES
BIBLIOGRAPHY	TERMINOLOGY	WORKED EXAMPLES	STEPS

CONTENTS **PAGE**

SECTION 1: POLICY

Policy Guidelines(outcomes): TRIGONOMETRY

A	B	C
(a) Definitions of the trigonometric ratios $\sin\theta$, $\cos\theta$ and $\tan\theta$ in a right-angled triangles. (b) Extend the definitions of $\sin\theta$, $\cos\theta$ and $\tan\theta$ to $0^\circ \le \theta \le 360^\circ$ (c) Derive and use values of the trigonometric ratios (without using a calculator for the special angles $\theta \in \{0^\circ; 30^\circ; 45^\circ; 60^\circ; 90^\circ\}$. (d) Define the reciprocals of trigonometric ratios.	(a) derive and use the identities: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \sin^2\theta = 1$. (b) derive the reduction formulae (c) determine the general solution and / or specific solutions of trigonometric equations (d) establish the sine, cosine and area rules	Proof and use of the compound angle and double angle identities. Solve problems in two and three dimensions applying the sine, cosine and area rules.
Solve problems in two dimensions.	Solve problems in 2-dimensions.	

(SOURCE: CURRICULUM AND ASSESSMENT POLICY STATEMET(CAPS) SENIOR PHASE GRADE 10-12 MATHEMATICS)

Concepts and mark allocation:

Concept(s)	Grade	Marks
Trigonometry – solving triangles and Identities	10,11 & 12	± 12
Trigonometry - Reduction Formula and Equations	10,11 & 12	± 18
Trigonometry Graphs	10 &11	± 12
Trigonometry 2D and 3D	10,11 & 12	± 8
TOTAL		**50**

Diagnostic report recommendations:

- When solving right angled triangles, use Pythagoras theorem to find all sides to apply trigonometric ratios; x, y and r.
- Teaching must explain the reduction of co-functions in relation to the quadrants.
- General solutions must be explained using the reference angle to find the value of the required angles. Explain when and why is the solution general and when not.
- Teaching must consider analysis and interpretation of graphs.
- Identify triangles with a diagram(text) and classify as follows:
    - ✓ Right angled- triangles – Trig ratios and Pythagoras
    - ✓ area rule – when the word area is mentioned
    - ✓ cosine rule – given three sides of a triangle
    - ✓ sine rule – any condition that does not satisfy the cosine rule.

T

	Word	Description / Definition	Symbols / Formula
1	Reduction Formulae	Are used to reduce any angle greater than 90^0 to an acute angle	$90^0 \pm \theta$ $180^0 \pm \theta$ $360^0 \pm \theta$
2	Co-function	Two angles are said to be complementary angles if their sum is equal to 90°.	$sin\,\theta = cos(90° - \theta)$ $cos\,\theta = sin(90° - \theta)$ $sin(90° + \theta) = cos\,\theta$ $cos(90° + \theta) = -sin\,\theta$
3	Trig-identity	Is a statement of equality with a left-hand side and can be proved by using certain identities which are derived from the basic trig-functions	$tan\,\theta = \frac{sin\,\theta}{cos\,\theta}$ $sin^2\,\theta + cos^2\,\theta = 1$ $sin^2\,\theta = 1 - cos^2\,\theta$ $cos^2\,\theta = 1 - sin^2\,\theta$
4	Restriction	An identity is true for all values of the variable except for those values the identity is not define for or undefined.	
5	General solution	To find all angles (also larger than 360^0 and smaller than 0^0) that will satisfy the equation	If it is a sine or cosine function, add $k.360°, k \in Z$and if it is a tangent function, add$k.180°, k \in Z$
6	Compound angles	The trigonometric identities $\cos(A - B) = \text{cosAcosB} + \text{sinAsinB}$ $\cos(A+B) = \text{cosAcosB} - \text{sinAsinB}$ $\sin(A - B) = \text{sinAcosB} - \text{sinBcosA}$ $\sin(A + B) = \text{sinAcosB} + \text{sinBcosA}$	
7	Double angles	Trigonometric ratio of $2x$ an angle	$\text{cos2A} = \cos^2 A - \sin^2 A$ $\text{cos2A} = 1 - 2\sin^2 A$ $\text{cos2A} = 2\cos^2 A - 1$ $\text{sin2A} = \text{2sinAcosA}$
8	Period	If a sine, cosine or tangent graph completes a full cycle, in a certain interval, measured in degrees .	sine and cosine graphs: $\frac{360^0}{b}$

9	Amplitude	It is $\frac{1}{2}$ distance between the maximum and minimum y-value of the graph and is always a positive value	
10	Solve a triangle	If a triangle is not right-angled, any sides or angles can be determined by the sine, cosine or area rules. The triangle can be solved if 3 values are given. sine − rule: s s ∠ ; ∠ ∠ s cosine − rule: s ∠ s ; s s s area − rule: s ∠ s	Sine rule: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ Cosine rule: $b^2 = a^2 + c^2 - 2.a.c.\text{CosA}$ Area rule: Area $\Delta\text{ABC} = \frac{1}{2}\text{ab SinC}$

SECTION 2: Trigonometry Ratios, Cartesian plane and Identities

Trigonometry Ratios and Identities

Content and worked examples:

There are three special ratios in the study of Trigonometry, namely the sine, cosine and tangent ratios. These ratios are applied to a right-angled triangle, they define the relationships between the sides and angles.

We can define the sine, cosine and tangent of an angle as follows:

1.

$$\text{sine of an angle } \theta = \frac{\text{length of the side opposite angle } \theta}{\text{length of the hypotenuse}}$$

$$\therefore \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

2.

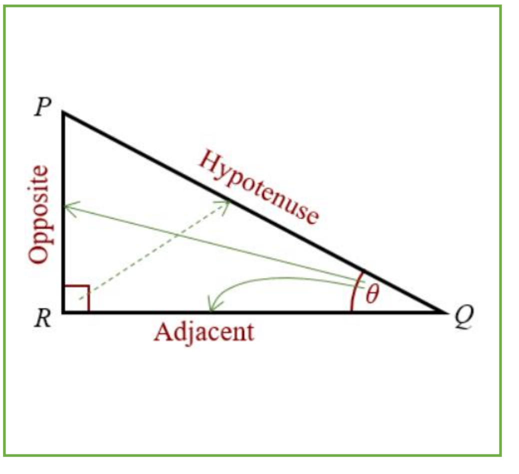
$$\text{cosine of an angle } \theta = \frac{\text{length of the side adjacent to angle } \theta}{\text{length of the hypotenuse}}$$

$$\therefore \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

3

$$\text{tangent of an angle } \theta = \frac{\text{length of the side opposite angle } \theta}{\text{length of the side adjacent to angle } \theta}$$

$$\therefore \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



Right-angled triangles

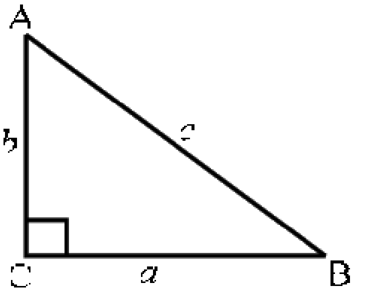
These triangles are fundamental to the study of trigonometry. In earlier grades you were introduced to the theorem of Pythagoras which describes the relationship between the three sides of a right-angled triangle.

From the Theorem of Pythagoras:

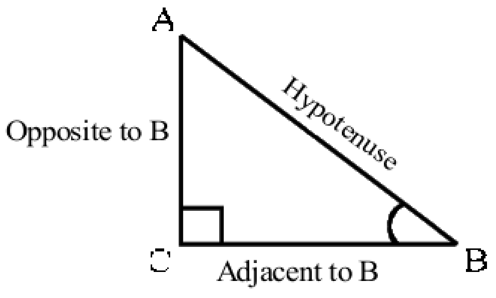
$c^2 = a^2 + b^2$

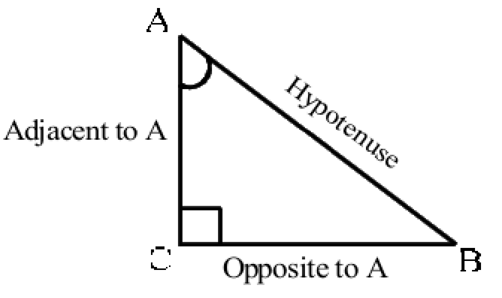
$a^2 = c^2 - b^2$

$b^2 = c^2 - a^2$



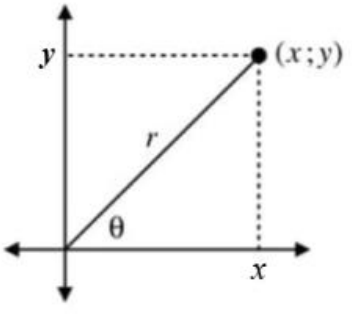
Furthermore, we will label the three sides as **adjacent, opposite and hypotenuse**. The longest side (opposite the right-angle) is called the hypotenuse. The opposite and adjacent sides are dependent on which angle is used as the reference point.





Cartesian Plane

In a cartesian plane the Trigonometric ratios are defines as:



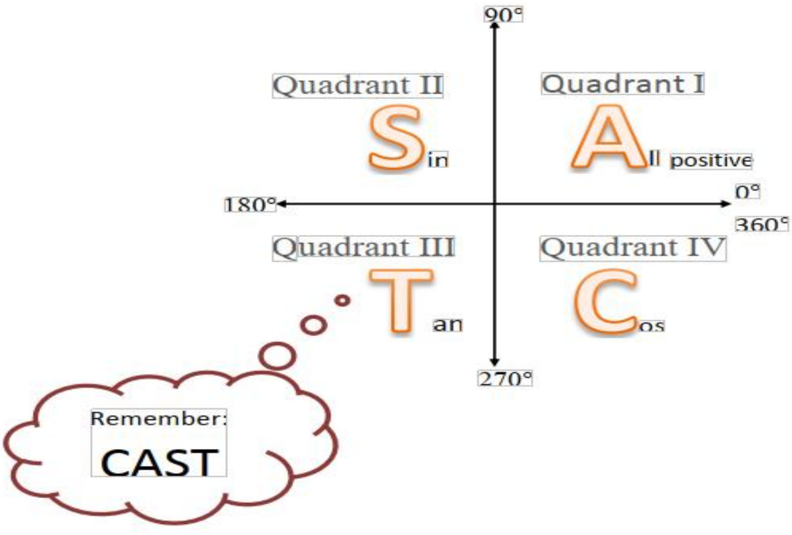
$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x}$

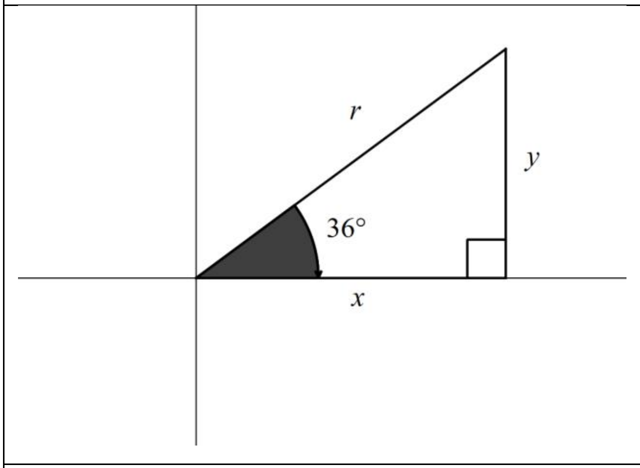
The angle formed will determine the sign of each trigonometric function in that quadrant.

- All trigonometric functions are positive in the first quadrant.
- sin θ is positive in the second quadrant and tan θ and cos θ are negative.
- tan θ is positive in the third quadrant and sin θ and cos θ are negative.
- cos θ is positive in the fourth quadrant and sin θ and tan θ are negative.

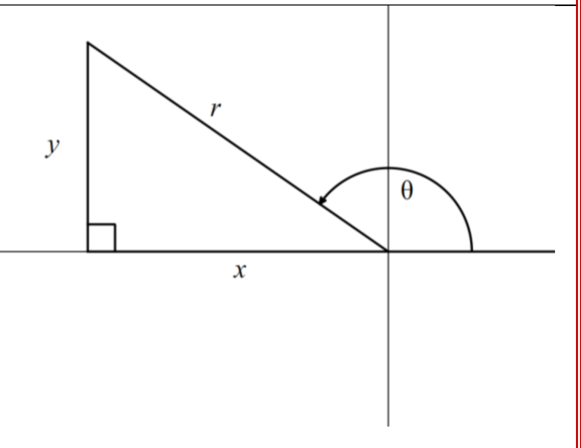


The following diagrams will show us how to draw our right-angled triangle in different quadrants.

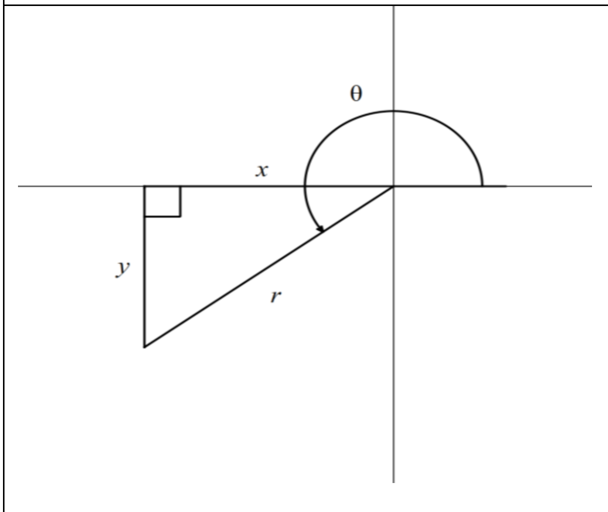
1st Quadrant



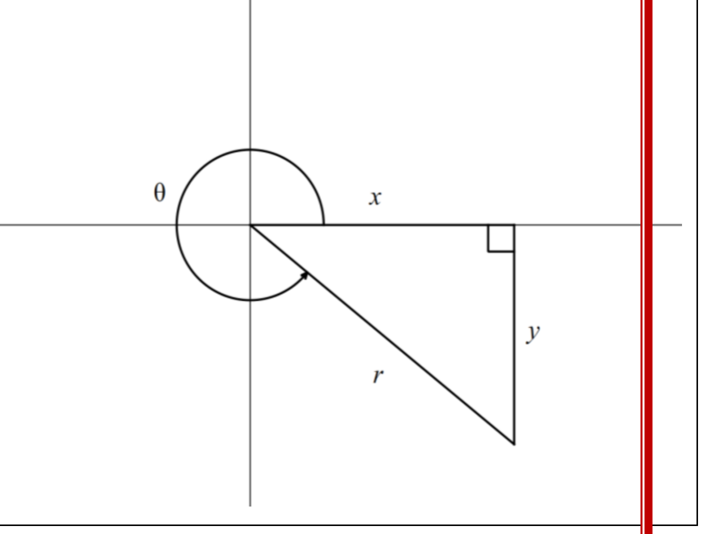
2nd Quadrant



3rd Quadrant



4th Quadrant



Worked example 1

Determine the quadrant which satisfies the following conditions and draw the accurate triangle.

a. $sin\ \theta < 0\ and\ \ 90° < \theta < 270°$.

b. $cos\theta > 0\ and\ tan\ \theta < 0$

c. $7\ tan\ \theta +\ 3 = 0\ and\ \theta\ \in\ [90°: 270°,]$

Solutions

a. $sin\ \theta\ <\ 0\ and\ 90° <\ \theta\ < 270°$.

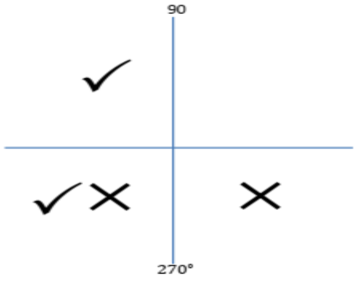
Step 1:

→use conditions to identify the quadrants and tick them with different symbols on the Cartesian plane.

Sin is negative in the 3rd and 4th quadrants.

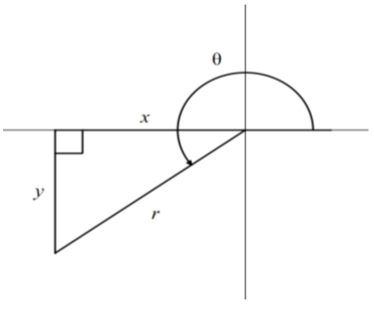
This interval represents 2nd and 3rd quadrants

Given that sin θ < 0° and 90° $<\ \theta\ < 270°$.



Where X represents the quadrant, we obtained from sin and tick is the quadrants obtained from the interval. We can notice the 3rd quadrant it satisfies both conditions, hence the terminal arm will be in the 3rd quadrant.

→ Complete the right-angled triangle.

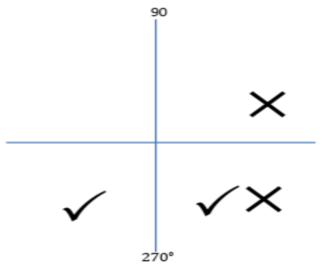


b. cos θ > 0 and tan θ < 0

cos is positive in the 1st and 4th quadrants.

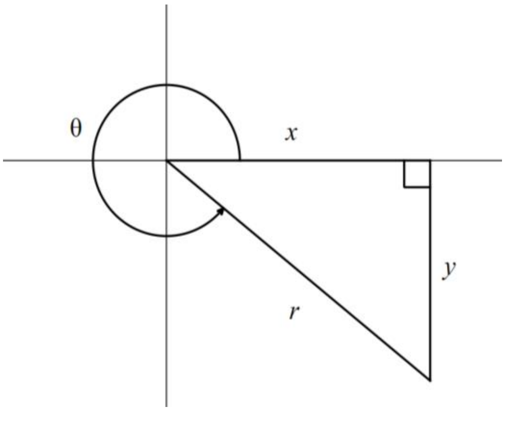
tan is negative in the 2nd and 4th quadrants

Given that cos θ > 0 and tan θ < 0.



Where X represents the quadrant, we obtained from cos and tick is the quadrants obtained from the tan. We can notice the 4th quadrant it satisfies both conditions, hence the terminal arm will be in the 4th quadrant.

→ Complete the right-angled triangle.



c. $7\ tan\ \theta + 3 = 0\ and\ \theta \in [90°: 270°,]$

Answer:

The first step is to simplify the first equation in such a way that we only have trig ratio (tan θ) on the left-hand side.

$7tan\ \theta + 3 = 0$

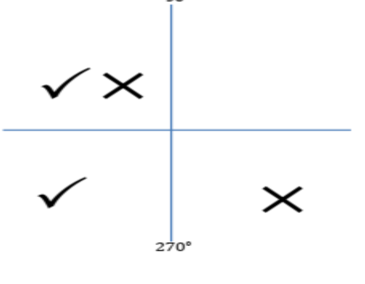
$7tan\ \theta = -3$

$tan\ \theta = \frac{-3}{7}$

→use conditions to identify the quadrants and tick them with different symbols on the Cartesian plane.

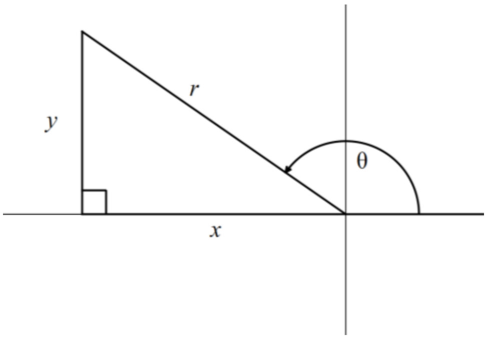
tan is negative in the 2nd and 4th quadrants.

This interval represents 2nd and 3rd quadrants



Given that $tan\ \theta = \frac{-3}{7}$ and θ ∈ [90°: 270°, Where X represents the quadrant, we obtained from tan and tick is the quadrants obtained from the interval. We can notice the 2nd quadrant satisfies both conditions, hence the terminal arm will be in the 2nd quadrant.

→ Complete the right-angled triangle.



For some trigonometry problems, it is helpful to draw a diagram showing the angle involved and the *x*, *y* and *r* values. It is advisable to find all the values of *x*, *y* and *r* before solving the trigonometry problems that will need the right-angled triangle: $x^2+y^2 = r^2$ where r is the hypotenuse.

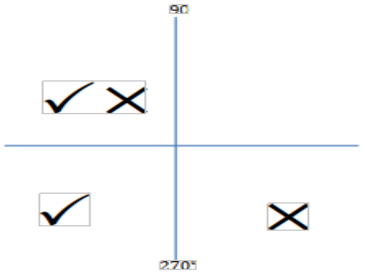
Remember hypotenuse is ALWAYS positive, and for the values of *x* and *y*, they can be positive or negative depending on the quadrant you are working in.

Worked example 2

Given that $sin\ \theta\ = \frac{3}{5}$, and θ ∈ [90°: 360°,] find the value of the following ***without the use of a calculator.***

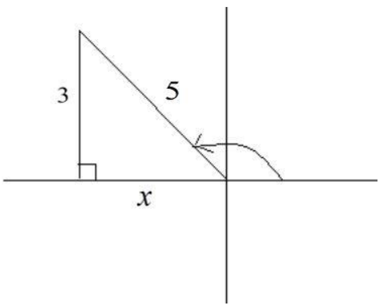
Solutions:

without a calculator – It means you must use the values from the TRIANGLE



We are going to draw the triangle in a 2nd quadrant.

$$sin\ \theta\ = \frac{opposite}{hypotenuse} = \frac{3}{5} = \frac{y}{r},$$



In the triangle above we do not have the value of x, then we will calculate it using **theorem of** Pythagoras:

$x^2 + y^2 = r^2$

$x^2 + (3)^2 = (5)^2$

$x^2 + 9 = 25$

$x^2 = 25 - 9$

$x^2 = 16$

$x = \sqrt{16}$

$x = \pm 4$

$x = -4$, because the x value in the 2nd quadrant is negative.

We can answer the following questions using the triangle above.

a. $cos\ \theta$
b. $tan(180° - \theta)$
c. $sin(-\theta)$
d. $sin\ (\theta - 180°)$
e. $cos\ (450° + 2\theta)$
f. $sin\ (45° - \theta)$

Solutions

a. $cos\ \theta = \frac{opposite}{hypotenuse}$

$= \frac{-4}{5}$

$= -\frac{4}{5}$

b. $tan(180° - \theta)$

$= -tan\ \theta = \frac{opposite}{adjacent}$

$= -(\frac{3}{-4})$

$= \frac{3}{4}$

c. $sin(-\theta)$

$= -sin\ \theta$

$= -\frac{3}{5}$

d. $sin\ (\theta - 180°)$

$= \sin(-180° + \theta)$

$= sin\ [-(180° - \theta)]$

$= \quad -sin\ (180° - \theta)$

$= \quad -sin\ \theta$

$= \ -\frac{3}{5}$

e. $cos\ (450° + 2\theta)$

$= \quad cos\ (450° - 360° + 2\theta)$

$= \quad cos\ (90° + 2\theta)$

$= \quad -\sin 2\theta$

$= \quad -2sin\theta . cos\theta$

$= \quad -2(\frac{3}{5}).(-\frac{4}{5})$

$= \quad \frac{24}{25}$

f. $sin\ (45° - \theta)$

$= sin\ (45° - \theta)$

$= sin\ 45°. cos\ \theta° - cos\ 45°. sin\ \theta°$

$= \frac{1}{\sqrt{2}} . -\frac{4}{5} - \frac{1}{\sqrt{2}} . \frac{3}{5}$

$= \frac{4}{5\sqrt{2}} - \frac{3}{5\sqrt{2}}$

$= \frac{1}{5\sqrt{2}}$

$= \frac{1}{5\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}}{10}$

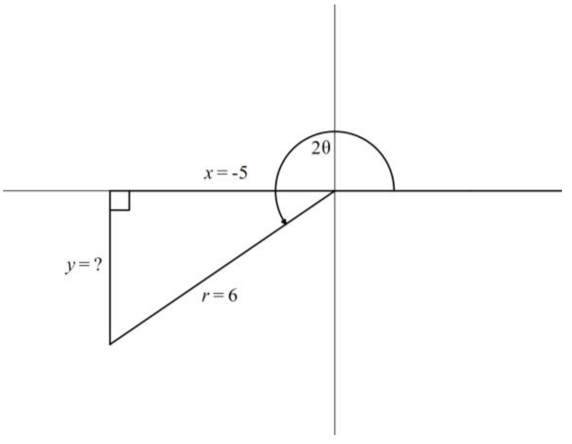
Worked example 3

If $\cos 2\theta = -\frac{5}{6}$, where $2\theta \in [180°; 270°]$, calculate, **without using a calculator**, the values in simplest form of:

1. $\sin\ 2\theta$
2. $\sin^2\theta$

Solutions

2θ is in 3rd quadrant because cos is negative, and the interval represents the 3rd quadrant.



Before we can answer the questions, we can note that in the triangle we do not have y-value.

$$x^2 + y^2 = r^2$$
$$(-5)^2 + y^2 = (6)^2$$
$$y^2 = 36 - 25$$
$$y^2 = 11$$
$$\text{y} = \sqrt{11}$$
$$y = \pm\sqrt{11}$$

$y = -\sqrt{11}$, because the y value in the 3rd quadrant is negative.

a. $Sin\ 2\theta = \frac{-\sqrt{11}}{6}$

b. $cos2\theta = -\frac{5}{6}$

$$-\frac{5}{6} = 1 - 2\ \sin^2\theta$$
$$2\ sin^2\theta = 1 + \frac{5}{6}$$
$$2\ sin^2\theta = \frac{11}{6}$$
$$sin^2\theta = \frac{11}{6} \div 2$$
$$sin^2\theta = \frac{11}{12}$$

Worked example 4

If $\cos 42° = p$ find the value of the following in terms of p.

a. Calculate the value of x, y and r.
b. Indicate all angles in a triangle
c. $sin\ 42°$
d. $cos\ 138°$
e. $tan\ 312°$
f. $sin\ 132°$
g. $cos\ (-48°)$
h. $sin\ 84°$
i. $sin\ 402°$
j. $cos\ 768°$
k. $2\ sin\ 21°. cos\ 21°$
l. $\cos^2 24° - \sin^2 24°$
m. $1 - 2sin^2\ 24°$
n. $sin\ 21°. cos\ 21°$
o. $cos\ 21°$
p. $sin\ 21°$

Solutions

a. $\sin 42° = \frac{p}{1} = \frac{y}{r}$

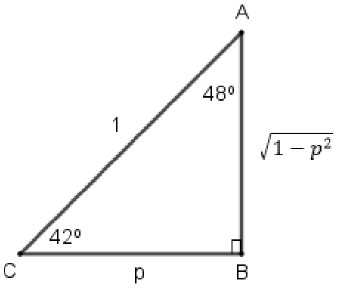
$x^2 + p^2 = 1^2$

$x^2 = 1 - p^2$

$x = \pm\sqrt{1 - p^2}$

$x = -\sqrt{1 - p^2}$

b.



c. $\sin 42°$

$= \frac{\sqrt{1-p^2}}{1}$

$= \sqrt{1 - p^2}$

d. cos 138°

$= \quad cos\,(180 \; - \; 42°)$

$= \quad -cos\,42°$

$= \quad -\,p$

e. tan 312°

$= \quad tan\,(\,360° \; - \; 48°)$

$= \quad -tan\,48°$

$= \quad -\frac{p}{\sqrt{1-p^2}}$ (Rationalize)

$= \frac{-p}{\sqrt{1-p^2}} \times \frac{\sqrt{1-p^2}}{\sqrt{1-p^2}}$

$= \frac{-p\sqrt{1-p^2}}{1-p^2}$

f. sin 132°

$= \quad sin\,(180° - 48°\,)$

$= \quad sin\,48°$

$= \quad p$

g. $cos\,(-48°)$

$= \quad cos\,48°$

$= \quad \sqrt{1-\,p^2}$

h. sin 84°

$= \quad sin\,2\,(42°)$

$= \quad 2sin\,42\;cos\,42°$

$= \quad 2.\sqrt{1-\,p^2}\;.p$

$= \quad 2p.\sqrt{1-\,p^2}$

i. sin 402°

$= \quad sin\,(402° - 360°)$

$= \quad sin\,42°$

$= \quad \sqrt{1-\,p^2}$

j. cos 768°

$= \quad cos\,(\,768° - 360° - 360°)$

$= \quad cos\,48°$

$= \quad \sqrt{1-\,p^2}$

k. 2 sin 21°.cos 21°

$= \quad sin\,2(21°)$

$= \quad sin\ 42°$

$= \quad \sqrt{1-\ p^2}$

l. $\cos^2 24° - \sin^2 24°$

$= \quad cos\ 2(24°)$

$= \quad cos\ 48°$

$= \quad \sqrt{1-\ p^2}$

m. $1 - 2\sin^2 21°$

$= \quad cos\ 2(21°)$

$= \quad \cos 42\ °$

$= \quad p$

n. $sin\ 21°. cos\ 21°$

$= \quad (\frac{1}{2} \times 2 \sin 21°. \cos 21°)$ Remember: $\frac{1}{2} \times 2 = 1$

$= \quad \frac{1}{2}\ sin2\ (21°)$

$= \quad \frac{1}{2}\ sin\ 42°$

$= \quad \frac{1}{2}\ \sqrt{1-\ p^2}$

o. $cos\ 21°$

$cos\ 42° = cos\ 2(21°) = 2cos^2 21° - 1$

$p = 2\cos^2 21° - 1$

$p + 1 = cos^2 21°$

$\frac{1}{2}(p+1) = cos^2 21°$

$\sqrt{\frac{1}{2}(p+1)} = cos\ 21°$

p. $\sin 21°$

$\cos 42° = \cos 2(21°) = 1 - 2\sin^2 21°$

$p = 1 - 2\sin^2 21°$

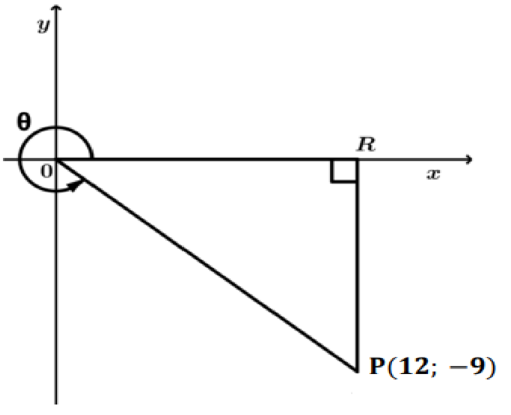
$2\sin^2 21° = 1 - p$

$\sin^2 21° = \frac{1}{2}(1-p)$

$\sin 21° = \sqrt{\frac{1}{2}(1-p)}$

Worked example 5

5.1 In the diagram below, P(12; −9) is a point in the Cartesian plane. Reflex angle $R\hat{O}P$ is equal to θ.



Calculate, **without using a calculator**, the:

5.1.1 Length of OP

5.1.2 The value of:

(a) $\cos(90° - \theta)$

(b) $\sin(\theta + 30°)$

Solutions

5.1.1	$OP = \sqrt{144 + 81}$ $= \sqrt{225}$ $= 15$
5.1.2 (a)	$\sin\theta = -\frac{9}{15} = -\frac{3}{5}$
5.1.2 (b)	$\sin(\theta + 30°) = \sin\theta\cos 30° + \cos\theta\sin 30°$ $= \left(-\frac{3}{5}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{12}{15}\right)\left(\frac{1}{2}\right)$ $= -\frac{3\sqrt{3}}{10} + \frac{6}{15}$ $= \frac{4 - 3\sqrt{3}}{10}$

Trigonometry Identity

Trigonometry Identity – an equality which is true for all values of an unknown variable, for which both sides of the identity are defined (so no zero denominators). It differs from a trigonometric equation which is true only for certain values of the unknown variable.

- To prove an identity, you must prove LHS = RHS

a) Square Identity **b) Quotient identity**

$$sin^2\theta + cos^2\theta = 1$$

$$sin^2\theta = 1 - cos^2\theta$$

$cos^2\theta = 1 - sin^2\theta$ $tan\,\theta = \frac{sin\,\theta}{cos\,\theta}$

$$tan\,2\,\theta = \frac{sin\,2\,\theta}{cos\,2\,\theta}$$

$$tan(\theta + \alpha) = \frac{sin(\theta + \alpha)}{cos(\theta + \alpha)}$$

Worked example 1

Simplify the expression using the identities proved before:

a. $(1 - sin\theta)\,(1 + sin\theta)$ b. $\cos^2\theta - 1$

c. $\tan^2\theta - \tan^2\theta\,.\sin^2\theta$

Solutions:

(a) $(1-\sin\theta)(1+\sin\theta)$

$= 1 - \sin^2\theta$

$= \cos^2\theta$ (Use the identity $1-\sin^2\theta = \cos^2\theta$)

(b) $\cos^2\theta - 1 = -\sin^2\theta$ (By manipulating $\sin^2\theta + \cos^2\theta = 1$ we can deduce this) OR

$\cos^2\theta - 1 = -(1-\cos^2\theta)$ Factorise (or Change of sign rule)

$= -\sin^2\theta$ Use the identity: $\sin^2\theta = 1 - \cos^2\theta$

(c) $\tan^2\theta - \tan^2\theta.\sin^2\theta$

$= \tan^2\theta(1-\sin^2\theta)$ Factorise by taking out the HCF

$= \frac{\sin^2\theta}{\cos^2\theta}(\cos^2\theta)$ If $\tan\theta = \frac{\sin\theta}{\cos\theta}$ then $\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$

$= \frac{\sin^2\theta}{\cos^2\theta}\left(\frac{\cos^2\theta}{1}\right)$

$= \sin^2\theta$

c) Compound angles

When two angles are added or subtracted to form a new angle, then a **Compound** or **a double angle** is formed.

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

To derive the other formulae the formula for $cos(\alpha-\beta)$ is used.

Derivation of $\boldsymbol{cos(\alpha+\beta) = cos\alpha\ cos\beta - sin\alpha\ sin\beta}$:

$cos(\alpha+\beta) = cos(\alpha-(-\beta))$

$cos(\alpha+\beta) = cos\alpha.cos(-\beta) + sin\alpha.sin(-\beta)$

$but\ cos(-\beta) = cos\beta\ and\ sin(-\beta) = -sin\beta$

$\therefore cos(\alpha+\beta) = cos\alpha.cos\beta - sin\alpha.sin\beta$

Derivation of $\boldsymbol{sin(\alpha+\beta) = sin\alpha\ cos\beta + cos\alpha\ sin\beta}$:

$sin(\alpha+\beta) = cos\ (90° - (\alpha+\beta))$

$sin(\alpha+\beta) = cos((90°-\alpha) - \beta)$

$sin(\alpha+\beta) = cos(90°-\alpha).cos\beta + sin(90°-\alpha)\ sin\beta$

$but\ cos(90°-\alpha) = sin\alpha\ and\ sin(90°-\alpha) = cos\alpha$

$\therefore sin(\alpha+\beta) = sin\alpha.cos\beta + cos\alpha.sin\beta$

Derivation of $\boldsymbol{sin(\alpha-\beta) = sin\alpha\ cos\beta - cos\alpha\ sin\beta}$:

$sin(\alpha-\beta) = sin(\alpha+(-\beta))$

$sin(\alpha-\beta) = sin\alpha.cos(-\beta) + cos\alpha.sin(-\beta)$

$but\ cos(-\beta) = cos\beta\ and\ sin(-\beta) = -sin\beta$

$\therefore sin(\alpha-\beta) = sin\alpha.cos\beta - cos\alpha.sin\beta$

Worked example 2

a. Find the value of $sin\ (45° + 30°)$ without the use of a calculator.

b. Expand the following using the compound angle formula $cos(60° - A)$.

c. Find the value of $cos\ 15°$ without using a calculator.

Solutions

a. $sin\ (30° + 45°)$

$= sin\ 30°.cos\ 45° + cos\ 30°.sin\ 45°$

$= \frac{1}{2}.\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}}$

$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$

$= \frac{1+\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}+\sqrt{6}}{4}$

b. $cos(60° - A)$.

$= cos\, A . cos 60° + sin\, A . sin 60°$

$= cos\, A . \frac{1}{2} + sin\, A . \frac{\sqrt{3}}{2}$

$= \frac{1}{2}\,(cos\, A + \sqrt{3}\, sin\, A)$

c. $sin\, 15°$

$= sin\,(45° - 30°)$

$= sin\, 45° . cos\, 30° - cos\, 45° . sin\, 30°$

$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} . \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$

Worked example 3

Simplify without the use of a calculator:

a. $cos\, 70° . cos\, 10° + sin\, 70° . sin 10°$

b. $sin\, 20° . cos\, 40° + cos\, 20° . sin\, 40°$

c. $cos\, 50° cos\, 40° - sin\, 50° sin\, 40°$

d. $cos\, 3\beta . cos\, \beta + sin\, 3\beta . sin\, \beta$

e. $cos\, 3\theta . sin\theta + sin\, 3\theta . cos\, \theta$

f. $sin\, 10° sin\, 20° - cos\, 20° \,cos\, 10°$

Solutions

$a.$ $cos\, 70° . cos\, 10° + sin\, 70° . sin 10°$

$= cos\, 70°\, cos\, 10° + sin 70°\, sin 10°$

$= cos(70° - 10°)$

$= cos\, 60°$

$= \frac{1}{2}$

$b.$ $sin\, 20° cos\, 40° + cos\, 20° sin\, 40°$

$= sin(20° + 40°)$

$= sin60°$

$= \frac{\sqrt{3}}{2}$

$c.\quad cos\ 50° cos\ 40° - sin\ 50° sin\ 40°$

$= \cos(50° + 40°)$

$= \cos 90°$

$= 0$

$d.\quad cos\ 3\beta . cos\ \beta\ + sin\ 3\beta . sin\ \beta$

$= cos(3\beta\ -\ \beta)$

$= cos\ 2\ \beta$

$e.\quad sin\ \theta\ cos3\theta\ + cos\ \theta\ sin3\theta$

$= sin(\theta\ -\ 3\theta)$

$= sin(-2\theta)$

$= -sin2\theta$

$f.\quad sin\ 20°. sin10° - \ cos\ 20°. cos\ 10°$

$= -\ cos20°\ cos10°\ +\ sin20°\ sin10°$

$= -\ (cos20°\ cos10°\ -\ sin20°\ sin10°)$

$= -\ cos(20°\ +\ 10°)$

$= -\ cos\ 30°$

$= -\ \frac{\sqrt{3}}{2}$

<u>D) Double angles</u>

$sin\ 2\ A = 2\ sin\ A\ cos\ A$

$cos\ 2\ A = cos^2\ A - sin^2\ A$

$= 1 - 2\ sin^2\ A$

$= 2\ cos^2\ A - 1$

The double angles are derived from the fact that $2\theta = \theta + \theta$

Derivation of $\boldsymbol{sin\ 2\theta\ = 2sin\ \theta . cos\ \theta}$

$sin\ 2\ \theta\ =\ sin(\theta + \theta)$

$=\ sin\theta . cos\theta\ +\ cos\theta . sin\theta$

$=\ 2sin\theta . cos\theta$

Derivation of $\boldsymbol{cos2\theta\ = cos^2\ \theta - sin^2\ \theta}$

$cos\ 2\theta\ =\ cos\ (\theta + \theta)$

$=\ cos\theta\ . cos\theta\ -\ sin\theta . sin\theta$

$= cos^2\ \theta - sin^2\ \theta$

Derivation of $\boldsymbol{cos2\theta} = \mathbf{2}\, \boldsymbol{cos^2\theta} - \mathbf{1}$

$cos\ 2\theta = cos\ (\theta + \theta)$

$= cos\theta\ .cos\theta - sin\theta.sin\theta$

$= cos^2\theta - sin^2\theta$

$= cos^2\theta - (1 - cos^2\theta)$

$= 2cos^2\theta - 1$

Derivation of $\boldsymbol{cos2\theta} = \mathbf{1} - \mathbf{2}\boldsymbol{sin^2\theta}$

$cos\ 2\theta = cos\ (\theta + \theta)$

$= cos\theta\ .cos\theta - sin\theta.sin\theta$

$= cos^2\theta - sin^2\theta$

$= 1 - sin^2\theta - sin^2\theta$

$= 1 - 2sin^2\theta$

Worked example 4

1. Use the **double angle formulae of sine** to expand the following:

$(a)\ sin\ 6\beta$

$(b)\ sin\ 70°$

$(c)\ sin\ 10\theta$

2. Use the **double angle formulae of cosine** to expand the following:

$(a)\ cos\ 14\beta$

$(b)\ cos\ 50°$

$(c)\ cos\ 3\theta$

Solution

$1.\ (a)\quad sin\ 6\beta$

$= sin[2\ (\boldsymbol{3\beta})\ ]$

$= 2\ sin\ 3\beta.cos\ 3\beta$

$(b)\quad sin\ 70$

$= sin[2\ (\mathbf{35}°)\ ]$

$= 2\ sin\ 35°.cos\ 35°$

$(c)\quad sin\ 8\theta$

$= sin[2\ (\boldsymbol{4\theta})\ ]$

$= 2\ sin\ 4\theta.cos\ 4\theta$

$2.\ (a)\ cos14\beta$

$= cos\ [2\ (\boldsymbol{7\beta})\ ]$

$= cos^2 7\beta - sin^2 7\beta$

(b) cos 50°

$= \quad cos\,[2\,(25)\,]$

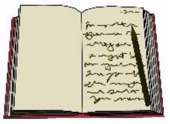
$= 2cos^2\,25 - 1$

$(c)\ \ cos\,3\theta$

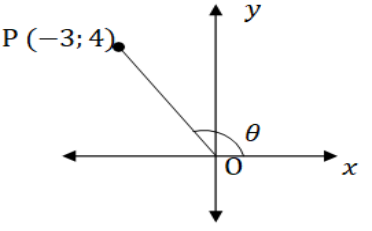
$= \ cos\,[2\,(\frac{3}{2}\theta)\,]$ $\qquad NB: 2 \times \frac{3}{2} = 3$

$= 1 - 2sin^2\left(\frac{3}{2}\theta\right)$

Activities



1. Use the following diagram to answer the questions below.



1.1 the length of OP.

1.2 $tan\theta$

1.3 $\sin{(180° - \theta)}$

1.4 $\cos{(180° + \theta)}$

1.5 $sin2\theta$

1.6 $cos2\theta$

1.7 $\sin{(30° + \theta)}$

1.8 $\cos{(45° - \theta)}$

2. If 5cos x = 4 and $x \in [90°; 360°]$, determine the value of 3sin 2x, without using a calculator.

3. If $13 \cos A = 5$ and $\tan B = \frac{3}{4}$, $A \in [0°; 270°]$ and $B \in [0°; 180°]$.

Determine, without using the calculator,:

3.1 $\sin A$

3.1 $\sin{(A + B)}$

4. 3 If $sin\ 27° = m$, express each of the following in terms of q.

4.1 $\sin 117°$

4.2 $tan\ (-27°)$

4.3 $\cos^2 63°$

4.4 $sin\ 72°$

4.5 $cos\ 54°$

5. Given $\sin 34° = p$, find the value of $sin17°cos17°$ in terms of p.

6. If $cos\ 73° cos\ 31° + sin\ 73° Sin\ 31° = \rho$, then determine the value of $\cos^2 21° - \sin^2 21° + 7$.

ADDITIONAL ACTIVITIES

7. Given $sin\ 24° = k$, determine the following in terms of k.

7.1 $tan\ 24°$

7.2 $sin\ 156°$

7.3 $cos\ 66°$

7.4 $sin\ 48°$

8. If $90° < A < 360°$ and $tan\ A = \frac{2}{3}$, determine without the use of a calculator:

8.1 $sin\ A$

8.2 $cos\ 2A - sin\ 2A$

9 Given that $sin\ x = t$, express the following in terms of t, without the use of calculator.

9.1 $cos\ (x - 90°)$

9.2 $sin\ 2x$

10 If $\sin 28° = a$ and $\cos 32° = b$, determine the following in terms of a and b.

10.1 $\cos 28°$

10.2 $\sin 118°$

11 If $\cos 58° = m$, determine the value of the following in terms of m, without the use of a calculator:

11.1 $\sin 572°$

11.2 $\tan 58°$

11.3 $\cos 64°$

11.4 $\sin 29° \cos 29°$

11.5 $\sin 2°$

SECTION 3: Reduction Formula

Reduction formula

Content and worked examples:

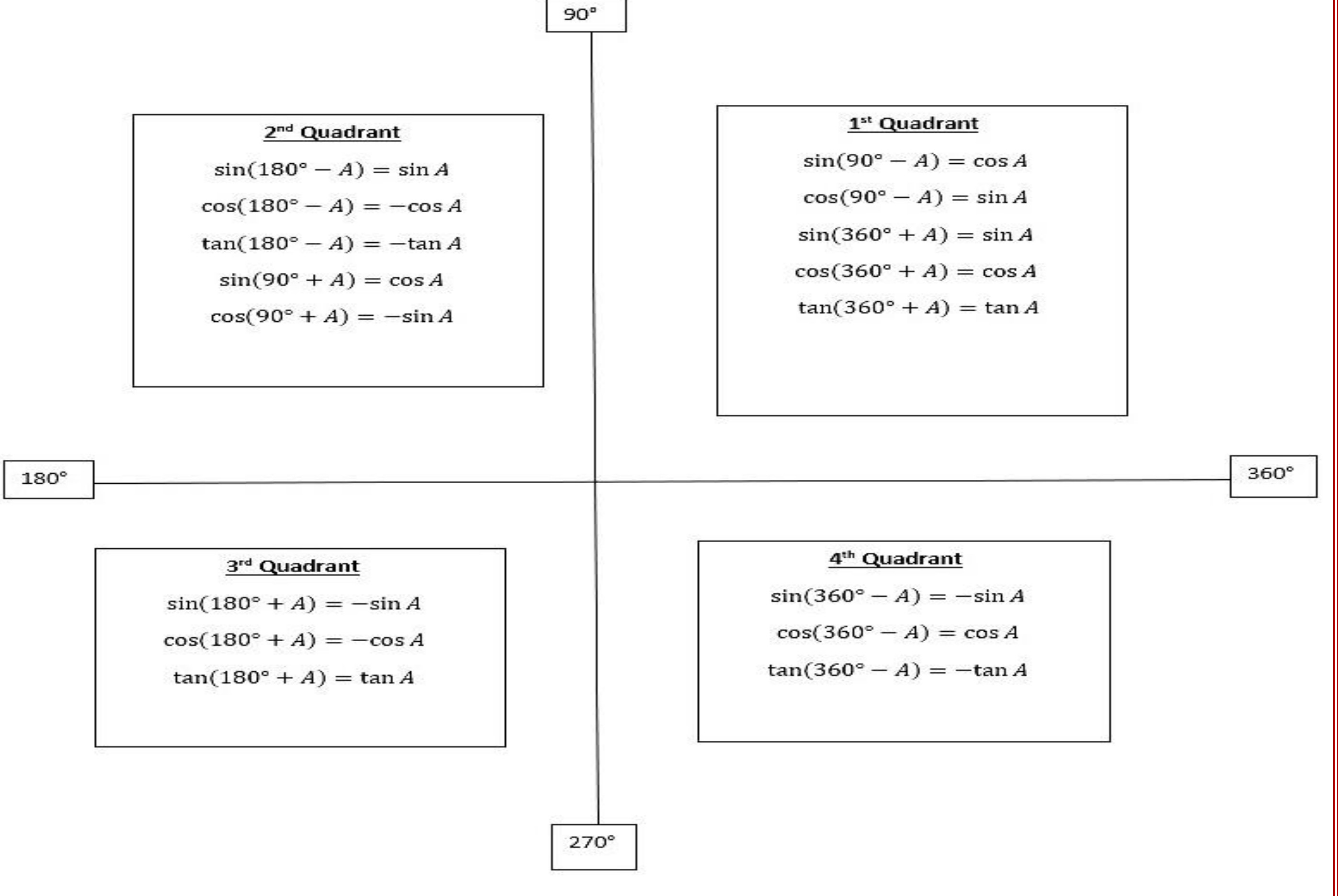
Reduction formulae are used to reduce the trigonometric ratio of any angle to the trigonometric ratio of an acute angle.

Approach to simplify exercises:

- ✓ Identify the quadrant to which the angle lie.
- ✓ If you use the reduction formula of $(90^0 \pm \theta)$, remember to change to co-function
- ✓ Write the correct trig ratio together with the acute angle.
- ✓ Take **one factor at a time** and make sense/simplify it.

  Note:

  - Angles greater than (360^0)
  - Negative angles
  - Complementary angles.



Worked example 1

Simplify as far as possible without the use of a calculator.

a. $\frac{\cos(180°+\theta)}{\cos(180°-\theta)}$

b. $\tan(90+\theta).\sin(180+\theta)$

c. $\sin 120°$

d. $\cos 260°$

e. $\frac{\tan(180°-\theta).\sin(360°+\theta)}{\tan(360°+\theta)}$

f. $\frac{\cos(360°-x).\sin(180°+x)}{\cos(180°+x)}$

Solution

a. $\frac{\cos(180°+\theta)}{\cos(180°-\theta)}$

$= \frac{\cos(\theta)}{-\cos(\theta)}$ (In 3rd quadrant and cosine is negative and 2nd quadrant and cosine is negative in the 2nd)

$= -1$

b. $\tan(90+\theta).\sin(180+\theta)$

$= \frac{\sin(90+\theta)}{\cos(90+\theta)} \times -\sin(\theta)$ (Change tan to $\frac{sin}{cos}$)

$= \frac{\cos(\theta)}{-\sin(\theta)} \times -\sin(\theta)$

$= \cos(\theta)$

c. $\sin 120°$ (Reduce obtuse angle to be acute angle)

$= \sin(180° - 60°)$

$= \sin 60°$

$= \frac{\sqrt{3}}{2}$

d. $cos 260°$ (Reduce obtuse angle to be acute angle)

$= \cos\,(180° + 80°)$

$= -cos\ 80°$

e. $\frac{\tan(180° - \theta).\sin(360° + \theta)}{\tan(360° + \theta)}$

3^{rd} 4^{th}

$$\frac{\tan(180° - \theta).\sin(360° + \theta)}{\tan(360° + \theta)}$$

4^{th}

$= \frac{(-\tan\theta)(+\sin\theta)}{\tan\theta}$ (Simplify)

$= -\sin\theta$

f. $\frac{\cos(360° - x).\sin(180° + x)}{\cos(180° + x)}$

$= \frac{(+\cos(x)).(-\sin(x))}{(-\cos(x))}$

$= \frac{-\sin(x)}{-1}$

$= \sin(x)$

Worked example 2

$a.\ sin(540° + \theta)$ $b.\ cos(720° - \theta)$

Solutions

a. $sin(540° + \theta)$

$= sin(540° + \theta - 360°)$

$= sin(180° + \theta)$

$= -sin\,\theta$

b. $cos(720° - \theta)$

$= cos(720° - \theta - 360°)$

$= cos(360 - \theta)$

$= cos\,\theta$

Negative angles

It is important to note that on the Cartesian plane an angle is considered to be positive if the rotation is in an anti-clockwise direction and negative if the rotation is clockwise.

$\boldsymbol{sin(-\theta) = -sin\,\theta}$

$\boldsymbol{cos(-\theta) = cos\,(\theta)}$

$\boldsymbol{tan(-\theta) = -tan\,\theta}$

Worked example 3

Write the following as a function value of a positive acute angle.

$a.\ sin\,(-50°)$ b. $\tan(150°)$ c. $cos\,(-35°)$

Solutions

a. $\sin(-50°)$

$= -sin\,50°$

b. $\tan(-150$

$= -tan\,150°$

$= -\tan(180° - 30°)$

$= -(-\tan 30°)$

$= tan30°$

c. $cos\,(-35°)$

$= cos\,35°$

Worked example 4

Remember:

a + b = b + a

-a + b = b – a

-a + b = - (a – b)

-a – b = - (a + b)

Simplify the following to a function value of θ:

$a.\ \ tan\ (\theta\ - 180°)$

$b.\ \ sin\ (-\ \theta\ -\ 360°)$

Solutions

a. $\tan(\theta - 180°)$

$= \ tan\ (-180° + \theta)$ [$tan\ (-\ angle) = -\ tan\ (angle)$]

$= -\ tan\ [-(180° - \theta)]$

$= \ -\ tan\ (180° - \theta)$

$= \ tan\ \theta$

b. $sin\ (-\ \theta\ -\ 360°)$ [take out common sign]

$= \ sin\ [-(\theta\ +\ 360°)]$

$= \ -sin\ (\theta\ +\ 360)\ °$ [$sin(-\ angle) = -\ sin\ (angle)$]

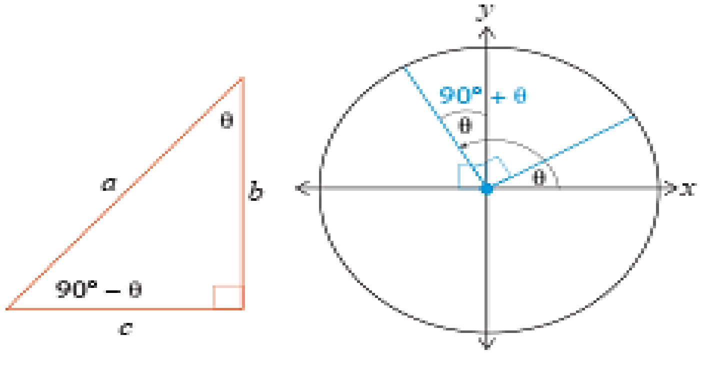
$= \ -sin\ (360° + \theta)$

$= \ -[-sin(\theta)]$

$= \ sin(\theta)$

The co-ratios are $90° - \theta$ and $90° + \theta$

Co-ratios are also called complementary ratios.



Look at the figure and you will see that :
$\sin(90° - \theta) = \frac{b}{a} = cos\theta$ and $\cos(90° - \theta) = sin\theta$

The $(90° + \theta)$ lies in Quadrant 2.

$\sin(90° + \theta) = \frac{b}{a}\cos\theta$ and $\cos(90° + \theta) = -sin\theta$, because $cos\theta$ is negative in Quadrant 2.

The following reduction formulae are used to reduce the co-functions:

$\sin(90° - \theta) = cos\theta$ $\cos(90° - \theta) = sin\theta$	$\sin(90° + \theta) = cos\theta$ $\cos(90° + \theta) = -sin\theta$

Worked example 5

(a) $\frac{sin\ (90° - \theta).sin\theta}{cos\theta.cos(90° + \theta)}$ (b) $sin(\theta - 90°)$ (c) $cos\ (\theta - 90°)$

Solution

(a) $\frac{sin\ (90° - \theta).sin\theta}{cos\theta.cos(90° + \theta)}$ $= \frac{cos\theta.sin\theta}{cos\theta.-sin\theta}$ $= -1$	(b) $sin(\theta\ - 90°)$ $= sin\ [-(\theta\ + 90°)]$ $= -sin(\theta\ +\ 90)$ $= -sin(90\ +\ \theta)$ $= -\ cos\ \theta$	(c) $cos\ (\theta - 90°)$ $=\ cos\ [-(\theta\ + 90°)]$ $= \cos(\theta\ +\ 90)$ $= cos\ (90\ +\ \theta)$ $= -sin\ \theta$

Worked example 6

a. Reduce $\frac{\tan 43°.\sin 47°.2\cos 137°}{2\cos 317°.\sin 133° - 1}$ to a single trigonometric ratio of one angle without the use of a calculator .

b. Show that $Sin(B + A) - Sin(B - A) = 2sinAcosA$

Solutions

a $$\frac{\tan 43°\,.\,\sin 47°\,.\,2\cos 137°}{2\cos 317°\sin 133° - 1}$$

$$\frac{\left(\frac{\sin 43°}{\cos 43°}\right)(\cos 43°)(-2\cos 43°)}{2\cos 43°\,.\,\cos 43° - 1}$$

$$\frac{\sin 43°(-2\cos 43°)}{\cos 2(43°)}$$

$$\frac{-\sin 2(43°)}{\cos 86°}$$

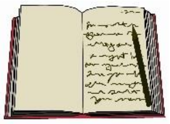
$$-\tan 86°$$

b $$\sin(B + A) - \sin(B - A)$$

$$= \sin B\cos A + \cos B\sin A - (sinB\cos A - \cos B\sin A)$$

$$= \cos B\sin A + \cos B\sin A$$

$$= 2\sin A\cos B$$



Activities:

1. Simplify the following **without using a calculator** to a single trigonometric ratio:

1.1 $\frac{sin(90^0+x).tan(360^0+x)}{sin(180^0+x).cos(90-x^0)+cos(540^0+x).cos(-x)}$

1.2 $\frac{cos(720^0-x).sin(360^0+x).tan(x-180^0)}{sin(-x).cos(90^0-x)}$

1.3 $\frac{cos(-210).sin^2 405^0 \, cos\, 14^0}{tan\, 120^0.sin\, 104^0}$

1.4 $\frac{sin(450^0-x)\, tan(x-180^0)\, sin\, 23^0.cos\, 23^0}{cos\, 46^0.sin(-x)}$

1.5 $\frac{4\, cos(-x).\cos(90^\circ+x)}{\sin(30^\circ-x).cos\, x+cos(30^\circ-x).sin\, x}$

2.1 **WITHOUT using a calculator**, determine the following in terms of sin 25°:

2.1.1 sin 335°

2.1.2 cos 50°

2.2 Simplify the following expression to ONE trigonometric ratio:

$$\frac{sin(-2x).(1-sin^2 x)}{sin(90^\circ+x).tan\, x}$$

2.3 **WITHOUT using a calculator**, simplify $(p\, tan\, 30^\circ + q\, sin\, 60^\circ)^2$.

2.4 Given: $cos(A-B) = cos\, A\, .\, cos\, B + sin\, A\, .\, sin\, B$

2.4.1 Use the formula for $cos(A-B)$ to derive a formula for $sin(A-B)$.

2.4.2 Prove that $sin\, 9A + sin\, A = 2\, sin\, 5A.cos\, 4A$.

2.4.3 Write down the maximum value of $3^{2\, sin\, 5A\, cos\, 4A}$

3. Evaluate

3.1 $sin\, 70^\circ\, cos\, 40^\circ - cos\, 70^\circ\, sin\, 40^\circ$

4. Evaluate each of the following without using a calculator.

4.1 Sin 75°

4.2 cos 15°

4.3 cos 105°

4.4 sin 165°

4.5 sin 36°cos 54°+cos 36°sin 54°

4.6 cos 42°cos 18° − sin 42°sin 18°

4.7 sin 85°sin 25°+cos 85°cos 25°

4.8 sin 70°cos 40° − cos 70°sin 40°

5 Proof the identity:

$$\frac{3\,sin\,x + 2\,sin\,2\,x}{2 + 3\,cos\,x + 2\,cos\,2\,x} = tan\,x$$

6. **Without the use of a calculator**, simplify the following expression to ONE trigonometric ratio:

$$2\,cos^2\,1\,5^\circ - 1 + \frac{2\,sin\,1\,40^\circ}{cos\,3\,10^\circ}$$

7 Simplify:

$$\frac{1 - sin(-\theta)\,cos(90^\circ + \theta)}{cos(\theta - 360^\circ)}$$

8 Consider: $\frac{cos\,2x + sin\,2x - cos^2\,x}{sin\,x - 2\,cos\,x} = -\,sin\,x$

8.1 Prove the above identity.

8.2 Determine the value of $\frac{cos\,2x + sin\,2x - cos^2\,x}{-3\,sin^2\,x + 6\,sin\,x\,cos\,x}$

9. Prove the following identity : $\frac{\sin 2x - cos\,x}{1 - cos\,2x - sin\,x} = \frac{1}{tan\,x}$

SECTION 4: Trigonometry Functions

In this section, we will discuss the graphs of the basic functions $y = sin\ \theta$ and $y\ =\ cos\ \theta$ and then introduce the general functions:

$y = asink(x - p) + q$, $y = a\cos k(x - p) + qy = a\,tan\,k\,(x - p) + q$

Worked example 1.

On the same system of axes, draw the sketch graphs of:

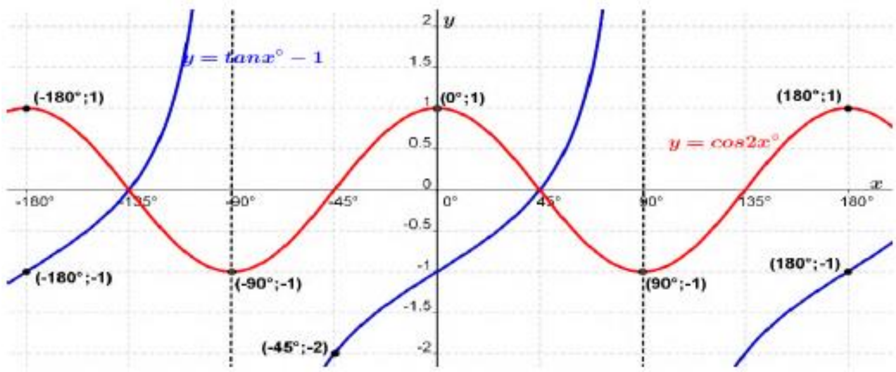
$f(x) = \tan x - 1$ and $g(x) = \cos 2x$, for the interval $[-180°; 0°]$. Show all the intercepts with the axes and the co-ordinates of the turning points. Show the asymptotes of $y = \tan x - 1$.

(a) Use the sketch graphs to determine the value of x if: $\cos 2x + 1 \leq \tan x$,in the interval $[-180°; 0°]$.

(b) If the curve of $y = \tan x - 1$ is moved upwards by 3 units, what will the new equation be?

(c) Write doen the period of $y = \cos 2x$.

Solution



(a) $\cos 2x + 1 \leq \tan x$

$\cos 2x \leq \tan x - 1$

$g(x) \leq f(x)$

$-135° \leq x < -90°$ and $-90° < x \leq 135°$

(b) $y = tan\ x + 2$

(c) period :180°

Worked example 2

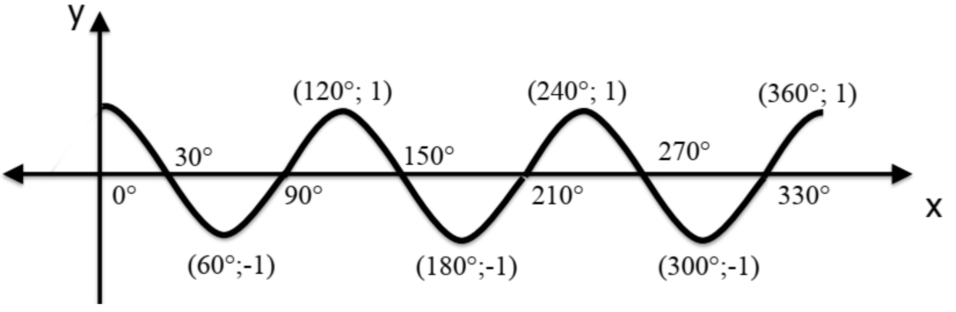
Find the period of f and draw the graphs of f:

a. $f(x) = \cos 3x$, $x \in [0°{:}360°]$

b . $f(x) = \tan\frac{1}{2}x$, $x \in [0°;360°]$

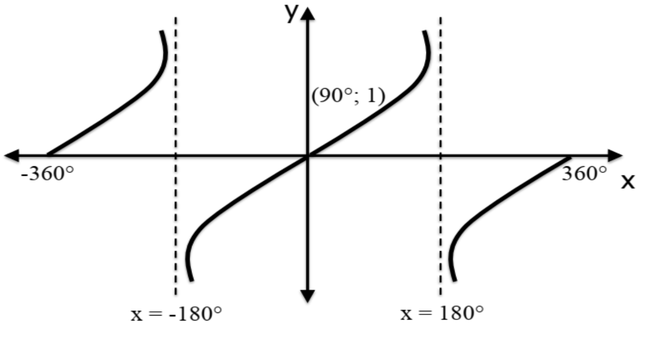
Solution

a. Period: $\frac{360°}{k} = \frac{360°}{3} = 120°$



We have three full waves of sine in the question.

b. Period: $\frac{180°}{k} = \frac{180°}{\frac{1}{2}} = 360°$



HORIZONTAL SHIFT

- $y = \sin(x - p)$ or $y = \cos(x - p)$ or $y = \tan(x - p)$

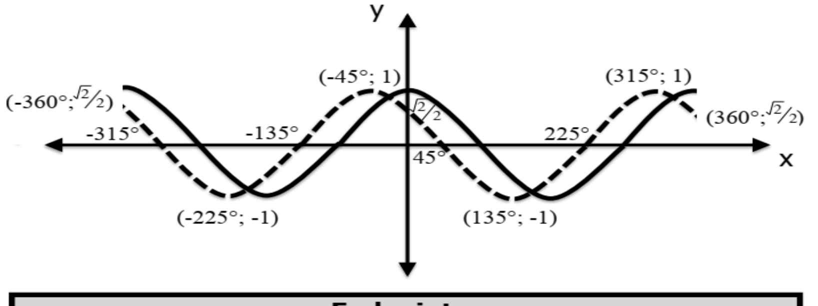
If p > 0 : shift right (e.g: $y = \sin(x - 30°)$)
p < 0 : shift left (e.g: $y = \cos(x + 45)$)

How to plot a horizontal shift:

- Plot the original curve
- Move the critical points left/right
- Label the x-cuts and turning points
- Calculate and label the endpoints and y-cut

Worked example 3

1. $y = \cos(x + 45°)$ for $x \in [-360°; 360°]$ (dotted line)
$y = \cos x$ (solid line - for comparison)

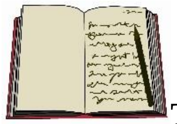


Endpoints:

$\cos(-360° + 45°) = \frac{\sqrt{2}}{2}$ and $cos(-360° + 45°) = \frac{\sqrt{2}}{2}$

y-cut:

$\cos(0° + 45°) = \frac{\sqrt{2}}{2}$

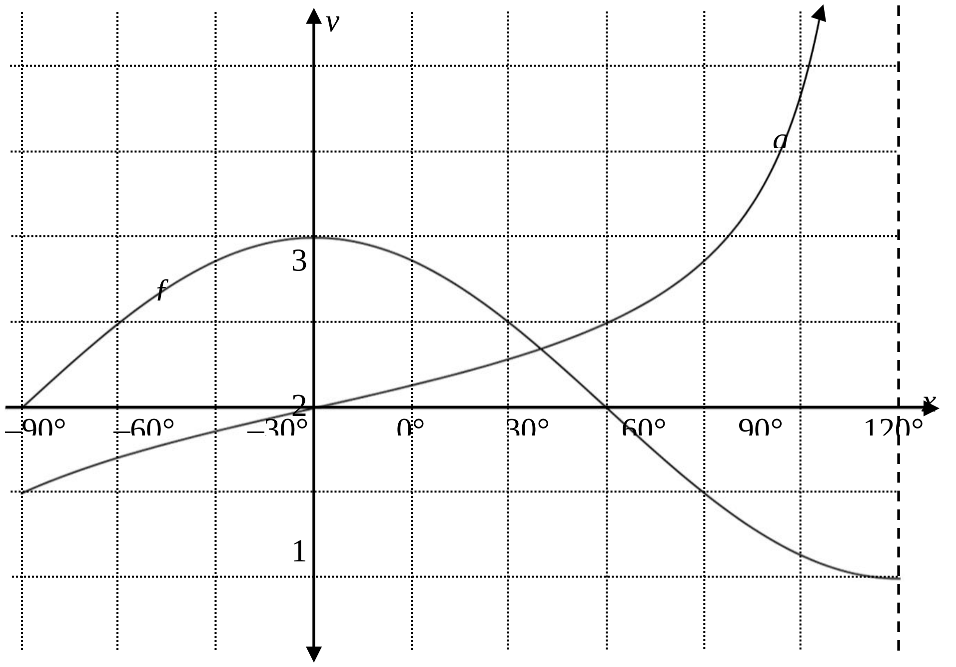


Trigonometry Functions

Activities:

1.

In the diagram below, the graphs of $f(x) = 2\cos x$ and $g(x) = \tan bx$ are drawn for the interval $x \in [-90° ; 120°]$.



Use the graphs to answer the following questions.

1.1 Write down the value of b.

1.2 Write down the range of g for the interval $x \in [-90°; 120°]$.

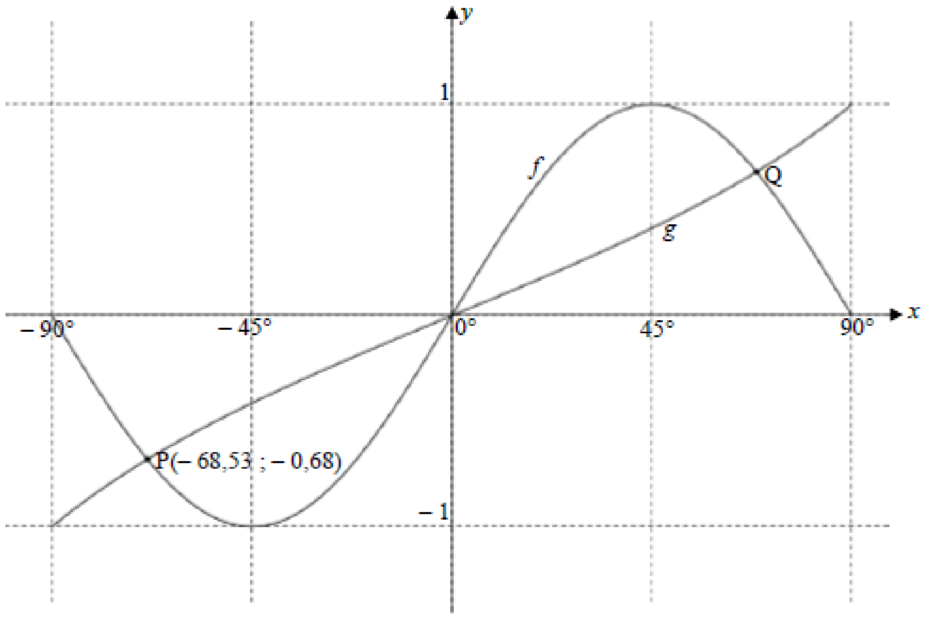
1.3 Write down the period of g.

1.4 Write down a value of x, in the given interval, where $g(x + 5°) - f(x + 5°) = 1$.

1.5 Write down TWO values of x in the given interval, where $\frac{g(x)}{f^{/}(x)}$ is undefined.

1.6 Write down the value of p if $\sum_{x=0°}^{p} 2\cos x = 0$

2 In the diagram below, the graphs of $f(x) = a\sin 2x$ and $g(x) = \tan bx$ for $x \in [-90°; 90°]$ are drawn. P $(-68{,}53 ; -0{,}68)$ and Q are the points of intersection of f and g.



2.1 Value of a

2.2 Coordinates of Q

2.3 x values of turning points of h, if $h(x) = f(x + 30°)$

2.4 Value(s) of x where $0{,}68 < g(x) \leq 1$

2.5 Value of m where $f(x + m) = -cos2x$

2.6 Value of b

3. Given : $f(x) = \cos\frac{x}{2}$ and $g(x) = \sin(x - 30°)$ for $x \in [-180°; 180°]$.

The graphs f and g intersect at A and B

3.1 Use the grid provided to draw sketch graphs of f and g on the same set of axes. Show clearly all the intercepts on the axes and the coordinates of the turning points.

3.2 Calculate the x coordinates of A and B.

3.3 State the period of graph f.

3.4 Determine the values of $x \in [-90°; 180°]$ for which $g(x).f(x) < 0$.

3.5 Given that the general solution of $f(x) = g(x)$ is: $x = 80° - k.240°, k \in z$. Determine the x values of point A and B.

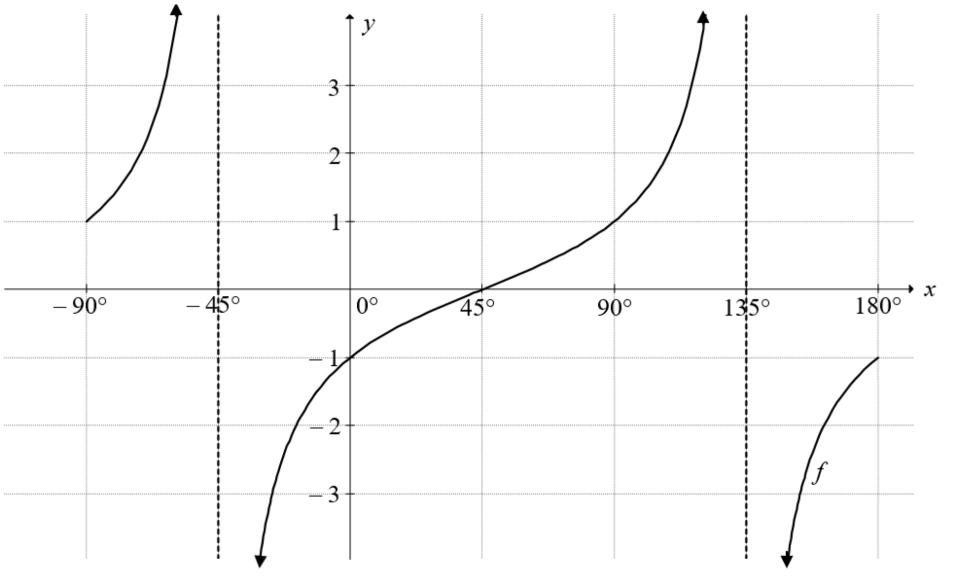
3.6 For which value(s) of x will.

3.6.1 $f(x) > g(x)$

3.6.2 $f'(x) . g(x) > 0$ where $x > 0°$

In the diagram below, the graph of $f(x) = tan(x - 45°)$ is drawn for $x \in [-90°; 180°]$.

4.



4.1 Write down the period of f.

4.2 Draw the graph of $g(x) = -\cos 2x$ for the interval $x \in [-90°; 180°]$ on the grid given in the ANSWER BOOK. Show ALL intercepts with the axes, as well as the minimum and maximum points of the graph

4.3 Write down the range of g.

4.4 The graph of g is shifted $45°$ to the left to form the graph of h. Determine the equation of h in its simplest form.

4.5.1 Use the graph(s) to determine the values of x in the interval $x \in [-90°; 90°]$ for which:

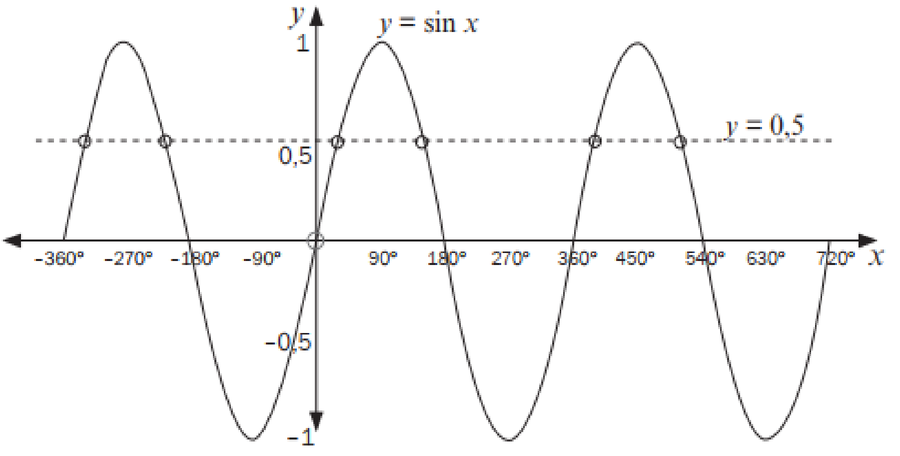
4.5.1 $f(x) > 1$

4.5.2 $2\cos 2x - 1 > 0$

SECTION 5: Trigonometric Equations

An equation with an unknown involving one or more trigonometric ratio of an unknown angle is called **trigonometric equation.** The solution of a trigonometric equation is the value of an unknown angle that satisfies the equation.

For example, $\sin\theta = \frac{1}{2}$, we know that θ could be 30°. However, there are other values for θ in other quadrants. Have a look at the graph for $\sin\theta = \frac{1}{2}, \theta \in [-360°; 720°]$ There are six values for θ between $-360°$ and $720°$



Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature).

Basic	Squares	Co-functions	Factorizing
• Isolate the trig ratio • Determine the reference angle • Choose the quadrant	• Do ALL four quadrants	• Sin and cos with different angles • The angle you change is the reference angle.	• Use the principles of Algebra.

Worked example

Given the equation $sinx = -\frac{\sqrt{3}}{2}$, find the general solution.

a. Solving with the aid of the CAST DIAGRAM

- Find the inverse of $sinx$ on both sides, to find the reference angle.

  [We always use the positive values].
  $x = sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60°$
  Reference angle (RA) = 60°

- Use the CAST diagram to check the quadrants where the given function is negative.

  Sine is negative in 3rd and 4th quadrants

- Find the general solution on both quadrants:

❖ The general solution for 3rd quadrants is $x = 180° + RA + k.360°, k \in \mathbb{Z}$
$x = 180° + 60° + k.360°$
$x = 240° + k.360°$

or

❖ The general solution for 4th quadrant is $x = 360° - RA + k.360°, k \in \mathbb{Z}$
$x = 360° - 60° + k.360°$
$x = 300° + k.360°$

b. Solve the general solution using graphical method.

The solution for $sinx = -\frac{\sqrt{3}}{2}$
Is

$x = sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + k.360°, k \in \mathbb{Z}$
$x = -60° + k.360°$

or

$x = 180° - sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + k.360° \quad , k \in \mathbb{Z}$
$x = 240° + k.360°$

Generally,

- if $sin\,\theta = a$, the general solution is given by:

  $\theta = sin^{-1}(a) + k.360°, k \in \mathbb{Z}$
  or
  $\theta = 180° - sin^{-1}(a) + k.360°, k \in \mathbb{Z}$

- If $cos\theta = a$, the general solution is given by:
  $\theta = \pm cos^{-1}(a) + k.360°, k \in \mathbb{Z}$

- If $\tan\theta = a$,the general solution is given by:

  $\theta = tan^{-1}(a) + k.180°, k \in \mathbb{Z}$

NB : Substitute "***a***" with its sign, if "***a***" is positive substitute as positive, if it is negative, substitute as negative

Worked example 1

Find the general solutions of the following equations:

a) $\cos\theta = -0.25$

$b)$ $\sin 2\theta = 0.766$

$c)$ $\cos(\theta - 15) = \frac{1}{2}$

$d)$ $3\sin(\theta - 15) + 1 = -0.456$

Solution

a) $\boldsymbol{\cos\theta = -0.25}$

$\theta = \pm\cos^{-1}(-0.25)$

$\theta = \pm 104.48 + k.360, \quad k \in Z$

b) $\boldsymbol{\sin 2\theta = 0.766}$

$reference\ angle = \sin^{-1}(0.766)$

$= 50$

$2\theta = 50 + k.360 \quad or \quad 2\theta = 180 - 50 + k.360 \ , \quad \boldsymbol{k} \in \boldsymbol{Z}$

$\theta = 25 + k.180 \quad or \quad \theta = 65 + k.180$

c) $\boldsymbol{\cos(\theta - 15) = \frac{1}{2}}$

$\theta = \pm\cos^{-1}\left(\frac{1}{2}\right)$

$\theta - 15 = \pm 60 + k.360, \quad k \in Z$

$\theta = 15 + 60 + k.360 \quad or \quad \theta = 15 - 60 + k.360$

$\theta = 75 + k.360 \quad or \quad \theta = -45 + k.360$

$\boldsymbol{d})$ $\boldsymbol{3\sin(\theta - 15) + 1 = -0.456}$

$\sin(\theta - 15) = -1.456$

$\sin(\theta - 15) = -\frac{182}{375}$

$reference\ angle = \sin^{-1}(-\frac{182}{375})$

$= -29.03$

$\theta - 15 = -29.03 + k.360 \quad or \quad \theta - 15 = 180 - (-29.03) + k.360 \ , \boldsymbol{k} \in \boldsymbol{Z}$

$\theta = -14.03 + k.360 \quad or \quad \theta = 209.03 + k.360$

Worked example 4

(a) For which value(s) of x will $\frac{2\tan x - \sin 2x}{2\sin^2 x}$ be undefined in the interval $0° \leq x \leq 180°$

(b) Prove the identity: $\frac{2\tan x - \sin 2x}{2\sin^2 x} = \tan x$

Solutions

a) The identity is undefined for

as: $2\sin^2 x = 0$

$\therefore\ \underline{\sin x = 0:}\quad x = 0°\ ;\ 180°$

or

$\underline{\tan x = \infty:}\quad x = 90°$

$\therefore\ x = 0°\ ;\ 90°\ ;\ 180°$

b) LHS $= \dfrac{2\tan x - \sin 2x}{2\sin^2 x}$

$= \dfrac{2\left(\dfrac{\sin x}{\cos x}\right) - 2\sin x\cos x}{2\sin^2 x}$

$= \left(\dfrac{2\sin x - 2\sin x\cos^2 x}{\cos x}\right) \times \dfrac{1}{2\sin^2 x}$

$= \dfrac{2\sin x(1-\cos^2 x)}{\cos x} \times \dfrac{1}{2\sin^2 x}$

$= \dfrac{2\sin x(\sin^2 x)}{\cos x} \times \dfrac{1}{2\sin^2 x}$

$= \dfrac{\sin x}{\cos x}$

$= \tan x$

$=$ RHS

Worked example 5

Determine the general solution of the following:

a. $6\ cosx - 5 = \frac{4}{cosx}$

b. $\sin 2x = cosx$

c. $3\sin x = 4cosx$

d. $4\cos^2 x + 4\sin x cosx + 1 = 0$

e. $\cos 2x - 4\sin x + 5 = 0$

f. $\sin x + cosx = 1$

g. $\sin 2x + cosx = 0$

h. $2\cos x = \cos(60 - x)$

i. $3\,sin^2 x - \sin 2x - \cos^2 x =$

Solutions

a. $6cos^2x - 5cosx - 4 = 0$ [Similar to the quadratic equation $6k^2 - 5k - 4 = 0$]

$(2cosx + 1)(3cosx - 4) = 0$

$(2cosx + 1) = 0$ or $(3cosx - 4) = 0$

$2cosx = -1$ or $3cosx = 4$

$cosx = -\frac{1}{2}$ $cosx \neq \frac{4}{3}$

$\therefore cosx = -\frac{1}{2}$

General solutions

$x = \pm cos^{-1}\left(-\frac{1}{2}\right) + k.360°, k \in \mathbb{Z}$

$x = \pm cos^{-1}\left(-\frac{1}{2}\right) + k.360°, k \in \mathbb{Z}$

$x = cos^{-1}\left(-\frac{1}{2}\right) + k.360°$ or $x = -cos^{-1}\left(-\frac{1}{2}\right) + k.360°$

$x = 120° + k.360°$ or $x = -120° + k.360°$

b. $\sin 2x = cosx, x \in [-90°, 180°]$

$\cos(90° - 2x) = cosx$ [The co-function is used]

$90° - 2x = \pm cos^{-1} cos(x) + k.360°, k \in \mathbb{Z}$

$90° - 2x = x + k.360°$ or $90° - 2x = -x + k.360°$

$-3x = -90° + k.360°$ $-x = -90° + k.360°$

$x = 30° - k.120°$ $x = 90° - k.360$

c. $3 \sin x = 4 \cos x$ [Divide both sides by $\cos x$ to create $\tan x$ on LHS]

$\frac{3 \sin x}{\cos x} = \frac{4 \cos x}{\cos x}$ [Trig identity for $\tan x$]

$3 \tan x = 4$

$\tan x = \frac{3}{4}$

Ref angle = 53,13°

$x = 53{,}13° + k180°\ k \in \mathbb{Z}$ (3)

d. $4\cos^2 x + 4\sin x \cos x + 1 = 0$

$4\cos^2 x + 4\sin x \cos x + (\sin^2 x + \cos^2 x) = 0$

$\sin^2 x + \cos^2 x = 1$

$5\cos^2 x + 4\sin x \cos x + \sin^2 x = 0$

$(5\cos x + \sin x)(\cos x + \sin x) = 0$

$5\cos x + \sin x = 0$ or $\cos x + \sin x = 0$

$\frac{5\cos x}{\cos x} = \frac{-\sin x}{\cos x}$ or $\frac{\cos x}{\cos x} = \frac{-\sin x}{\cos x}$

$5 = -\tan x \therefore \tan x = -5$ $\quad$ $1 = -\tan x \therefore \tan x = -1$

Reference angle = 78,69° $\quad$ Reference angle = –45°

$x = 180° - 78{,}69° + k180°$ or $\therefore x = 180° - 45° + k180°$

$\therefore x = 101{,}3° + k180°$ $\quad$ $\therefore x = 135° + k180° \quad k \in \mathbb{Z}$

e. $\cos 2x - 4\sin x + 5 = 0$

$1 - 2sin^2 x - 4\sin x + 5 = 0$ $\quad$ [$\cos 2x = 1 - 2sin^2 x$]

$-2sin^2 x - 4\sin x + 6 = 0$

$sin^2 x - 2\sin x - 3 = 0$ $\quad$ [Divided by -2]

$(\sin x - 3)(\sin x + 1) = 0$ $\quad$ [Factorize]

$\sin x \neq 3$ or $\sin x = -1$

$x = 90 + k.360, k \in Z$

f. $\sin x + cosx = 1$

$(\sin x + cosx)^2 = 1^2$

$sin^2 x + 2\sin x.\cos x + cos^2 x = 1$

$sin^2 x + cos^2 x + 2\sin x.\cos x = 1$ $\quad$ [$sin^2 x + cos^2 x = 1$]

$1 + 2\sin x.\cos x = 1$

$\sin 2x = 0$ $\quad$ [$2\sin x.\cos x = \sin 2x$]

$2x = 90° + k.360, k \in Z$

$x = 45° + k.180$ $\quad$ [divide EVERY TERM by 2]

g. $\sin 2x + \sin x = 0$

$2\sin x\cos x + \sin x = 0$ $\quad$ [Change double angles to single angles: sin 2*x*= 2sin*x*.cos*x*]

$\sin x\,(2\cos x + 1) = 0$ $\quad$ [Factorise by taking out the common factor of sin *x*]

$2\cos x + 1 = 0$ or $\sin x = 0$

$\cos x = -\frac{1}{2}$ $\quad$ $x = 0° + n.180°$

$x = \pm 120° + k.360°, \ k \in \mathbb{Z}$

h. $2\cos x = \cos(60° - x)$

$2\,cosx = \cos 60°.cosx + sin60°.sinx$

$2 = \cos 60° + \frac{\sin 60°.\sin x}{\cos x}$ $\quad$ [Divide everything by cos *x*]

$2 = \frac{1}{2} + \frac{\sqrt{3}}{2}.\tan x$

$\frac{3}{2} = \frac{\sqrt{3}}{2}.\tan x$

$\frac{3}{\sqrt{3}} = tanx$

Reference angle $= 60°$

$x = 60 + k.180°, \qquad k \in \mathbb{Z}$

ACTIVITIES

1. Determine the general solution for the following:
   1.1 $\sin^2 x - \sin x = \cos^2 x$
   1.2 $\sin^2 x + \sin x \cos x = 0$
2. Determine the general solution of : $3cos2x = 1 + 5cosx$
3. Solve for x: $4\sin^2 x - 3sinx - 1 = 0$ for $x \in [0°; 360°]$
4. Determine the general solution of $tan\, x = 2\, sin\, 2\, x$ where $\cos x < 0$.
5. Calculate the value of x, if $x \in [-180°; 360°]$ $cos2x = cosx + 2$

SUBJECT: MATHEMATICS

GRADE 12

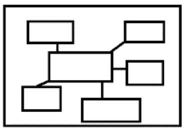
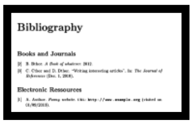
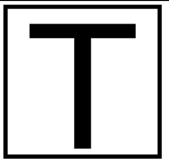
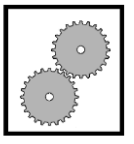
TERM 2

TEACHER AND LEARNER CONTENT MANUAL

Topic

EUCLIDEAN GEOMETRY

ICON DESCRIPTION

 MIND MAP	 EXAMINATION GUIDELINE	 CONTENTS	 ACTIVITIES
 BIBLIOGRAPHY	 TERMINOLOGY	 WORKED EXAMPLES	 STEPS

CONTENTS **PAGE**

SECTION 1: Examination guideline

ACCEPTABLE REASONS

THEOREM STATEMENT	ACCEPTABLE REASON(S)
LINES	
The adjacent angles on a straight line are supplementary.	∠s on a str line
If the adjacent angles are supplementary, the outer arms of these angles form a straight line.	adj ∠s supp
The adjacent angles in a revolution add up to 360°.	∠s round a pt **OR** ∠s in a rev
Vertically opposite angles are equal.	vert opp ∠s =
If AB || CD, then the alternate angles are equal.	alt ∠s; AB || CD
If AB || CD, then the corresponding angles are equal.	corresp ∠s; AB || CD
If AB || CD, then the co-interior angles are supplementary.	co-int ∠s; AB || CD
If the alternate angles between two lines are equal, then the lines are parallel.	alt ∠s =
If the corresponding angles between two lines are equal, then the lines are parallel.	corresp ∠s =
If the cointerior angles between two lines are supplementary, then the lines are parallel.	coint ∠s supp
TRIANGLES	
The interior angles of a triangle are supplementary.	∠ sum in Δ **OR** sum of ∠s in Δ **OR** Int ∠s Δ
The exterior angle of a triangle is equal to the sum of the interior opposite angles.	ext ∠ of Δ
The angles opposite the equal sides in an isosceles triangle are equal.	∠s opp equal sides
The sides opposite the equal angles in an isosceles triangle are equal.	sides opp equal ∠s
In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.	Pythagoras **OR** Theorem of Pythagoras
If the square of the longest side in a triangle is equal to the sum of the squares of the other two sides then the triangle is right-angled.	Converse Pythagoras **OR** Converse Theorem of Pythagoras
If three sides of one triangle are respectively equal to three sides of another triangle, the triangles are congruent.	SSS
If two sides and an included angle of one triangle are respectively equal to two sides and an included angle of another triangle, the triangles are congruent.	SAS **OR** S∠S
If two angles and one side of one triangle are respectively equal to two angles and the corresponding side in another triangle, the triangles are congruent.	AAS **OR** ∠∠S
If in two right angled triangles, the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of the other, the triangles are congruent	RHS **OR** 90°HS

THEOREM STATEMENT	ACCEPTABLE REASON(S)
The line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side	Midpt Theorem
The line drawn from the midpoint of one side of a triangle, parallel to another side, bisects the third side.	line through midpt || to 2[nd] side
A line drawn parallel to one side of a triangle divides the other two sides proportionally.	line || one side of Δ **OR** prop theorem; name || lines
If a line divides two sides of a triangle in the same proportion, then the line is parallel to the third side.	line divides two sides of Δ in prop
If two triangles are equiangular, then the corresponding sides are in proportion (and consequently the triangles are similar).	||| Δs **OR** equiangular Δs
If the corresponding sides of two triangles are proportional, then the triangles are equiangular (and consequently the triangles are similar).	Sides of Δ in prop
If triangles (or parallelograms) are on the same base (or on bases of equal length) and between the same parallel lines, then the triangles (or parallelograms) have equal areas.	same base; same height **OR** equal bases; equal height

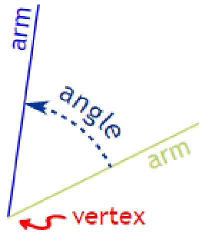
THEOREM STATEMENT	ACCEPTABLE REASON(S)
Equal chords in equal circles subtend equal angles at the circumference of the circles.	equal circles; equal chords; equal ∠s
Equal chords in equal circles subtend equal angles at the centre of the circles.	equal circles; equal chords; equal ∠s
The opposite angles of a cyclic quadrilateral are supplementary	opp ∠s of cyclic quad
If the opposite angles of a quadrilateral are supplementary then the quadrilateral is cyclic.	opp ∠s quad supp **OR** converse opp ∠s of cyclic quad
The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.	ext ∠ of cyclic quad
If the exterior angle of a quadrilateral is equal to the interior opposite angle of the quadrilateral, then the quadrilateral is cyclic.	ext ∠ = int opp ∠ **OR** converse ext ∠ of cyclic quad
Two tangents drawn to a circle from the same point outside the circle are equal in length	Tans from common pt **OR** Tans from same pt
The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.	tan chord theorem
If a line is drawn through the end-point of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.	converse tan chord theorem **OR** ∠ between line and chord

QUADRILATERALS	
The interior angles of a quadrilateral add up to 360°.	sum of ∠s in quad
The opposite sides of a parallelogram are parallel.	opp sides of ||m
If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.	opp sides of quad are ||
The opposite sides of a parallelogram are equal in length.	opp sides of ||m
If the opposite sides of a quadrilateral are equal , then the quadrilateral is a parallelogram.	opp sides of quad are = **OR** converse opp sides of a parm
The opposite angles of a parallelogram are equal.	opp ∠s of ||m
If the opposite angles of a quadrilateral are equal then the quadrilateral is a parallelogram.	opp ∠s of quad are = **OR** converse opp angles of a parm
The diagonals of a parallelogram bisect each other.	diag of ||m
If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.	diags of quad bisect each other **OR** converse diags of a parm
If one pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.	pair of opp sides = and ||
The diagonals of a parallelogram bisect its area.	diag bisect area of ||m
The diagonals of a rhombus bisect at right angles.	diags of rhombus
The diagonals of a rhombus bisect the interior angles.	diags of rhombus
All four sides of a rhombus are equal in length.	sides of rhombus
All four sides of a square are equal in length.	sides of square
The diagonals of a rectangle are equal in length.	diags of rect
The diagonals of a kite intersect at right-angles.	diags of kite
A diagonal of a kite bisects the other diagonal.	diag of kite
A diagonal of a kite bisects the opposite angles	diag of kite

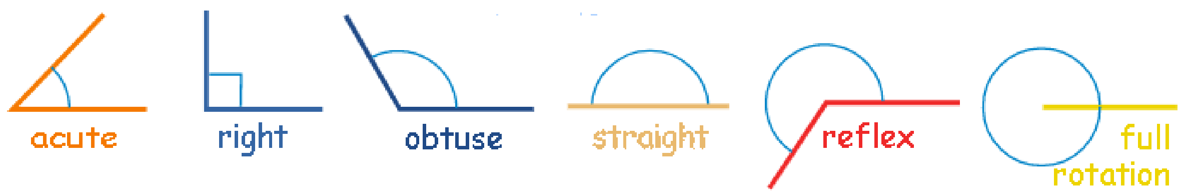
SECTION 2: LINES, ANGLES AND TRIANGLES

Definitions

Angles: an angle is "the amount of turn between two lines around their common point (the vertex)" – www.mathisfun.com



Types of angles

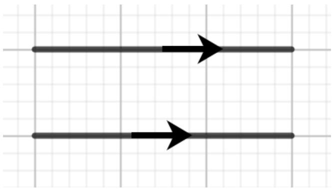


(www.mathisfun.com)

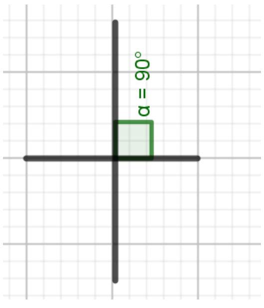
- Acute angle → an angle between 0° and 90°
- Right angle → an angle equal 90°
- Obtuse angle → an angle between 90° and 180°
- Straight angle → an angle equal 180°
- Reflex angle → an angle between 180° and 360°
- Full rotation (Revolution) → an angle equal 360°

Lines

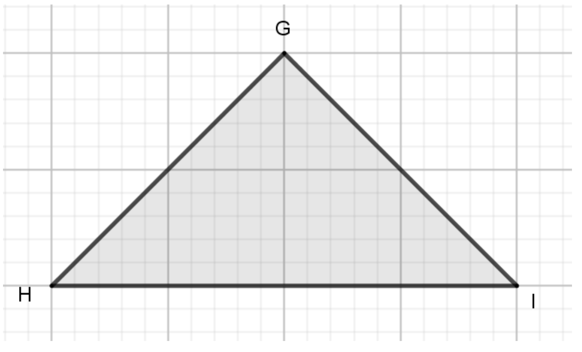
Parallel lines → lines on the same plane and that can never intersect as they are extended



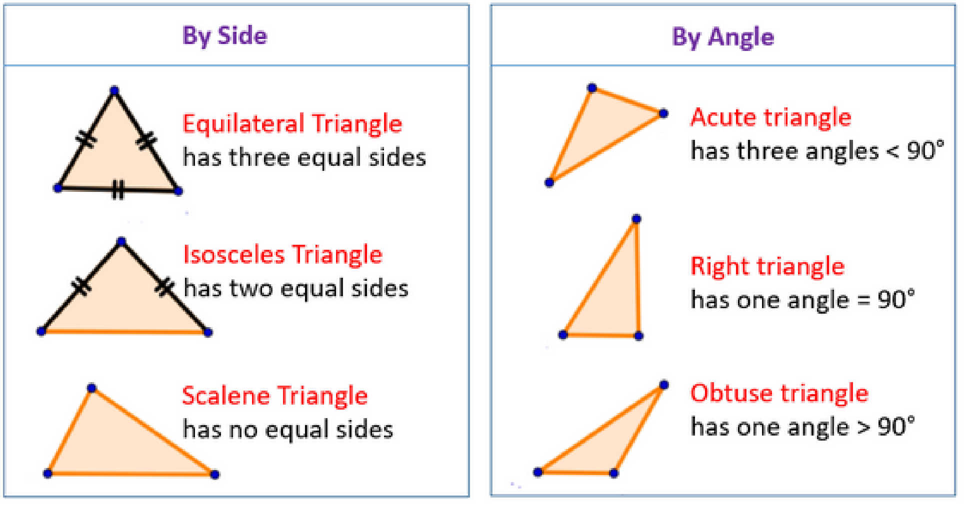
Perpendicular lines → lines that intersect at right angles (90°)



Triangles: a triangle is shape made of 3 straight lines, the shape is closed. It has 3 sides and 3 vertices.

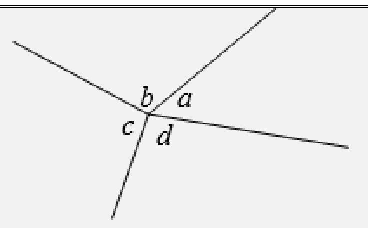
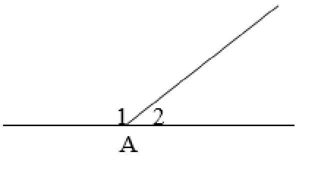
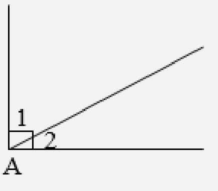
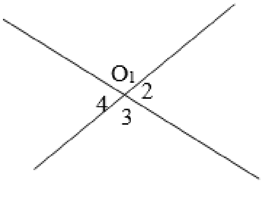
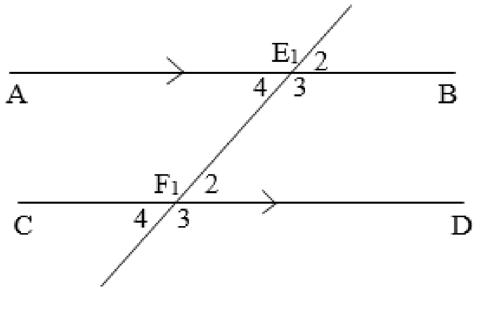


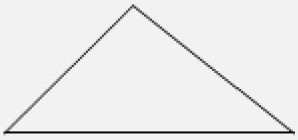
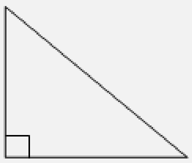
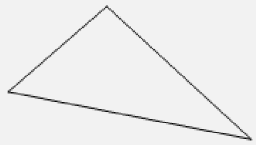
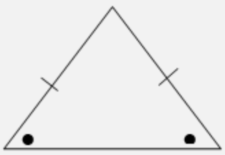
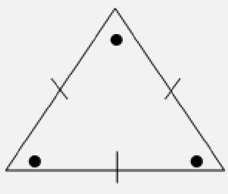
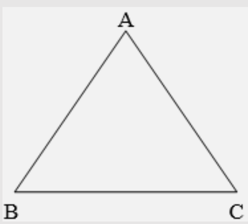
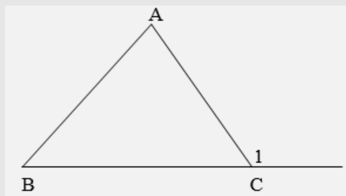
Types of triangles

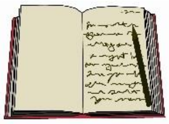


(www.curemath.com)

N.B All the theorem statements were taken from 2021 grade 12 mathematics examination guidelines.

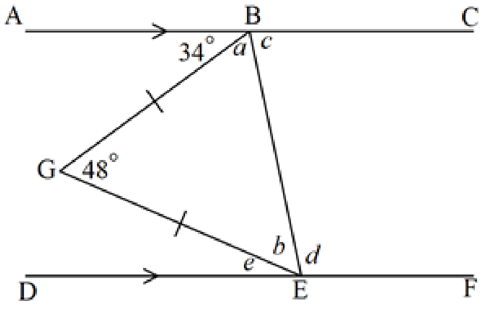
REVISION OF BASIC RESULTS ESTABLISHED IN EARLIER GRADES	
Intersecting lines	
1.1 Angles around a point add up to 360°. (Revolution)	 $a + b + c + d = 360^o$
1.2 The sum of angles on a straight line is 180°. Two angles that add up to 180° are said to be **supplementary**.	 $\hat{A}_1 + \hat{A}_2 = 180^o$
1.3 Two angles are **complementary** if their sum is 90° or a **right angle**.	 $\hat{A}_1 + \hat{A}_2 = 90^o$
1.4 Whenever two straight lines intersect, the **vertically opposite angles** are equal in size.	 $\hat{O}_1 = \hat{O}_3$ and $\hat{O}_2 = \hat{O}_4$
Parallel lines	
➢ **Alternate angles are equal** $\hat{E}_3 = \hat{F}_1$ and $\hat{E}_4 = \hat{F}_2$ ➢ **Corresponding angles are equal** $\hat{E}_1 = \hat{F}_1$; $\hat{E}_2 = \hat{F}_2$; $\hat{E}_3 = \hat{F}_3$ and $\hat{E}_4 = \hat{F}_4$ ➢ **Co-interior angles are supplementary** $\hat{E}_4 + \hat{F}_1 = 180^o$ and $\hat{E}_3 + \hat{F}_2 = 180^o$	

Types of triangles	
Acute-angles triangle	**Obtuse-angled triangle**
➢ All three angles are acute  NB: Acute angle is between 0^o and 90^o	➢ One obtuse angle and two acute angles  NB: Obtuse angle is between 90^o and 180^o
Right-angled triangle	**Scalene triangle**
➢ One right angle and two acute angles  NB: Right angle is equal 90^o	➢ All three sides have different lengths 
Isosceles triangle	**Equilateral triangle**
➢ Exactly two sides are equal in length ➢ The angles opposite equal sides are equal1  ➢ The perpendicular drawn from the vertex A of an isosceles Δ to the opposite side bisects the opposite side and the vertex angle.	➢ All three sides are equal in length ➢ All three angles are equal, i.e. each angles is 60°.  ➢ The perpendicular drawn from a vertex to the opposite side, bisects the vertex angle and the opposite side.
Some general properties of triangles	
❖ The sum of interior angles in a triangle is 180^o.  $\hat{A}+\hat{B}+\hat{C}=180^o$	❖ The exterior angle of a triangle is equal to the sum of the interior opposite angles.  $\hat{C}_1=\hat{A}+\hat{B}$

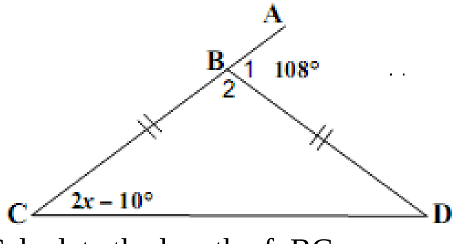


Activity

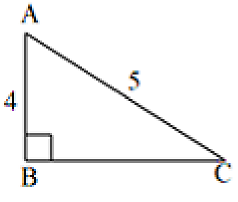
1. Find the size of the angles a, b, c, d and e



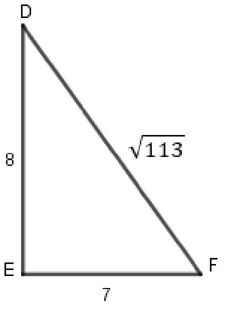
2. Calculate x, if $\hat{B}_1 = 108°$ and $\widehat{B}_2 = 2x - 10°$



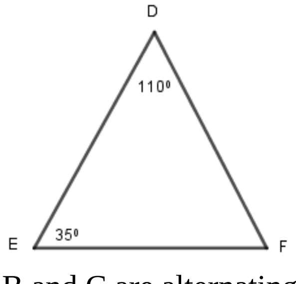
3. Calculate the length of BC



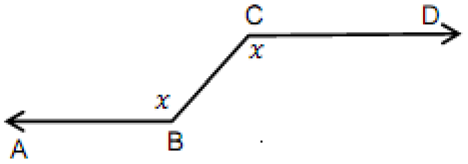
4. Prove that triangle ΔDEF is a right-angled triangle. Show which angle is = 90^o



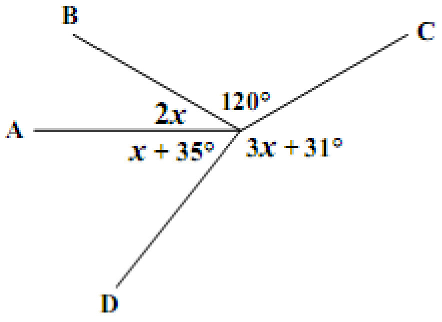
5. Prove that DE = DF



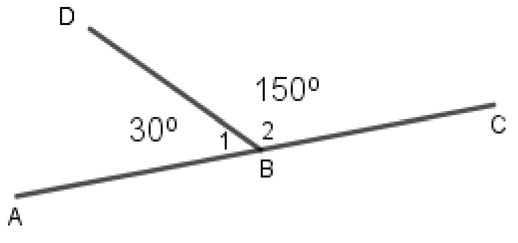
6. B and C are alternating angles equal to x. Is AB || CD? Give a reason for your answer.



7. Calculate x



8. Given that: $A\hat{B}D = 30^o$ and $D\hat{B}C = 150^o$. Prove, giving reasons, that ABC is a straight line

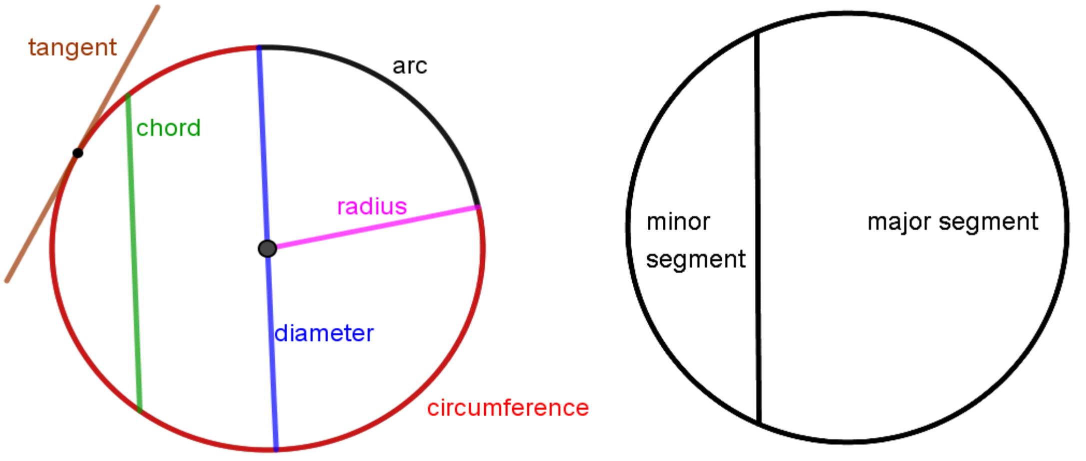


9. 9.1 If the corresponding angles between two lines are equal, then the lines are …………………..

9.2 If the ………………..angles between two lines are supplementary, then the lines are parallel.

SECTION 3 : Circle geometry

Different parts of a circle:



Definitions

- A **circle** is a set of points that are equidistant from a fixed point called the **center.**
- The **circumference** of the circle is the distance around the edge of a circle.
- The **radius** is a line from the centre to any point on the circumference of the circle.
- A **chord** divides the circle into two **segments**.
- A **diameter** is a chord that passes through the centre. It is the longest chord and is equal to twice the radius.
- An **arc** is part of the circumference.
- A **semi-circle** is half the circle.
- A tangent is a line touching the circle at a point.
- **Cyclic quadrilaterals** have all their vertices on the circumference of a circle.

Classical method: This is done by writing two columns (HOLY-CROSS), the first being a list of statements and the second column a matching list of legal justifications. i.e. theorems and axioms referred to as REASONS.

STATEMENT	REASON
1. $\hat{A}_2 = \hat{C}_1 = 35°$	Alt $\angle's$, $AB \parallel CD$
2.$M\hat{A}P + 35° = 75°$	Ext $\angle$ of Δ

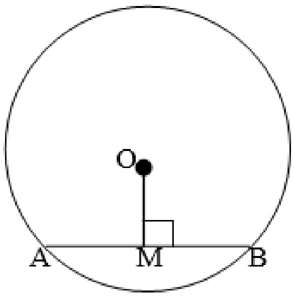
Circle theorems are divided into four groups :

1. Theorems centered around the center.

A.

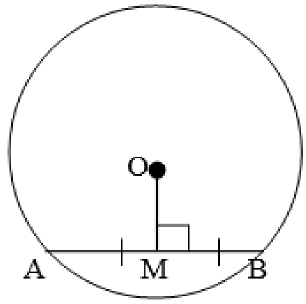
<table>
<tr>
<td>The line drawn from centre of the circle perpendicular to a chord bisects the chord.

<u>(line from centre ⊥ to chord)</u>



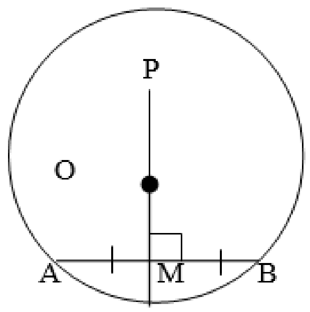
If OM ⊥ AB, then
AM = MB</td>
<td>The line segment joining the center of the circle to the midpoint of a chord is perpendicular to the chord.

<u>(line from center to midpoint of chord)</u>



If AM = MB, then
OM ⊥ AB</td>
<td>The perpendicular bisector of a chord passes through the center of the circle.

<u>(perp bisector of chord)</u>

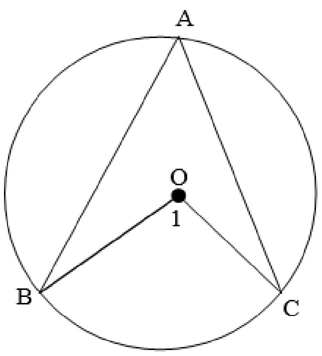


If OP ⊥ AB and AM = MB,
then PM passes through O</td>
</tr>
</table>

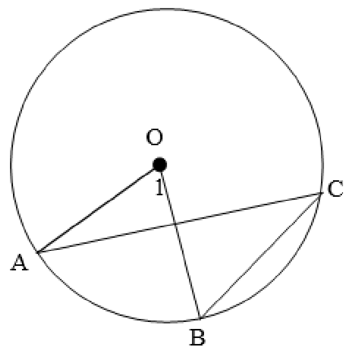
B.

The angle which an arc of a circle subtends at the center of the circle is twice the angle it subtends at the circumference of the circle.

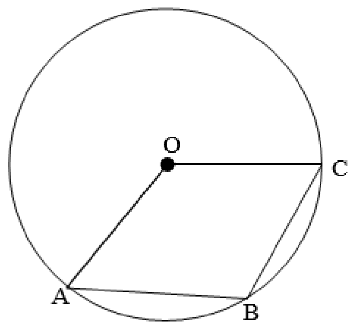
(∠at center = 2 × ∠ at circle)



$\hat{O}_1 = 2\ \hat{A}$



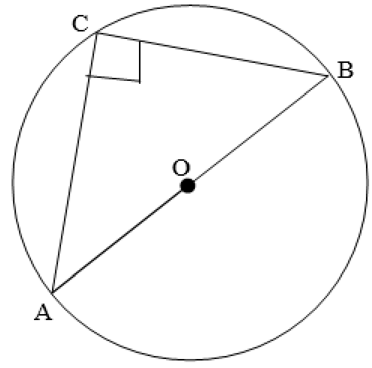
$\hat{O}_1 = 2\ \hat{C}$



Reflex $A\hat{O}C = 2\ \hat{B}$

C. The angle subtended at the circle by the diameter is a right angle.

<u>(∠s in semi-circle)</u>

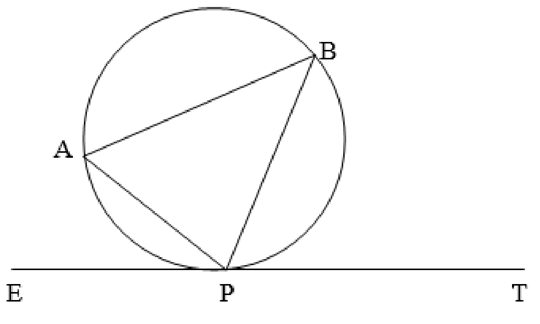


$\hat{C} = 90°$ (AB is a diameter)

2. Theorems centered around the tangent.

A. The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.

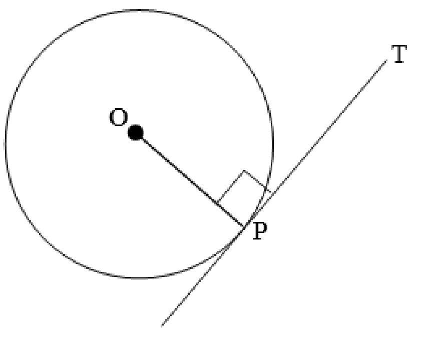
<u>(tan-chord theorem)</u>



$B\hat{P}T = \hat{A}$ and $A\hat{P}E = \hat{B}$

B. A tangent to a circle is perpendicular to the radius at the point of contact.

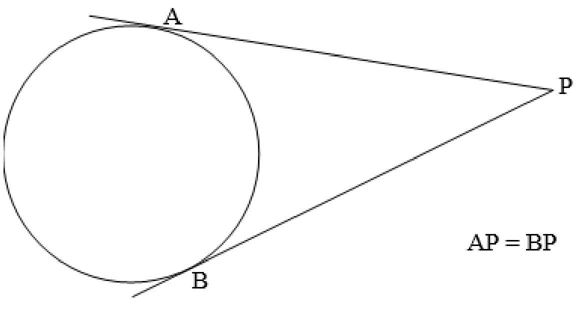
<u>(tan ⊥ radius)</u>



$PT \perp OP$

Two tangents drawn to a circle from the same point outside the circle are equal in length

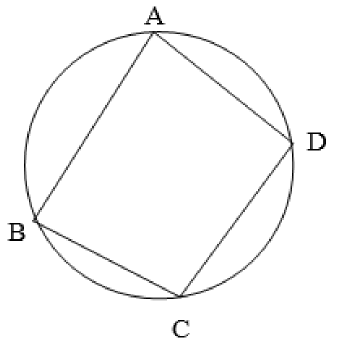
(tan. from same pt.)



3. Theorems centered around the cyclic quad.

A. The opposite angles of a cyclic quadrilateral are supplementary.

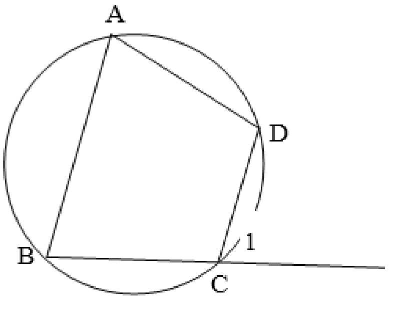
(opp ∠'s of cyclic quad)



$\hat{A} + \hat{C} = 180°$ and $\hat{B} + \hat{D} = 180°$

B. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

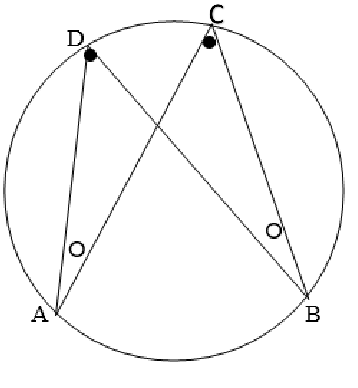
(ext. ∠ of cyclic quad.)



$\hat{C}_1 = \hat{A}$

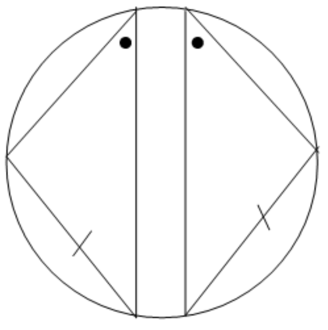
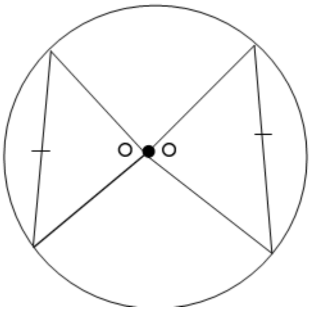
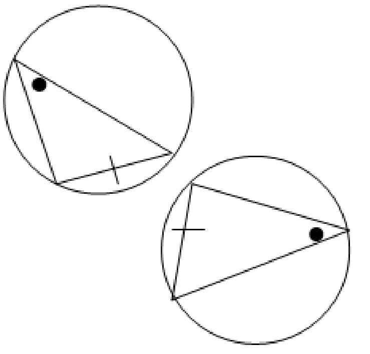
C. Angles subtended by a chord of the circle, on the same side of the chord, are equal.

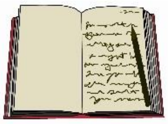
(∠'s in the same seg)



$\hat{D} = \hat{C}$ and $\hat{A} = \hat{B}$

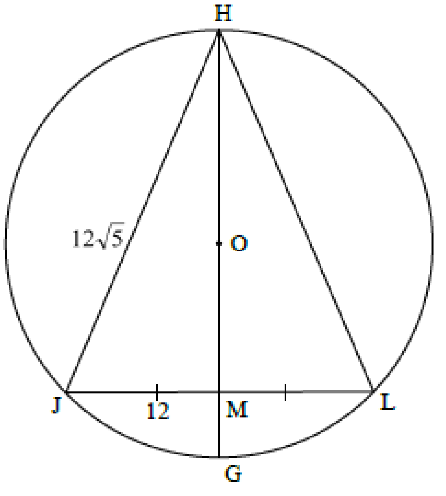
4. Equal chords corollaries

Equal chords subtend equal angles at the circumference of the circle.	Equal chords subtend equal angles at the centre of the circle	Equal chords, in equal circles subtend equal angles.
		



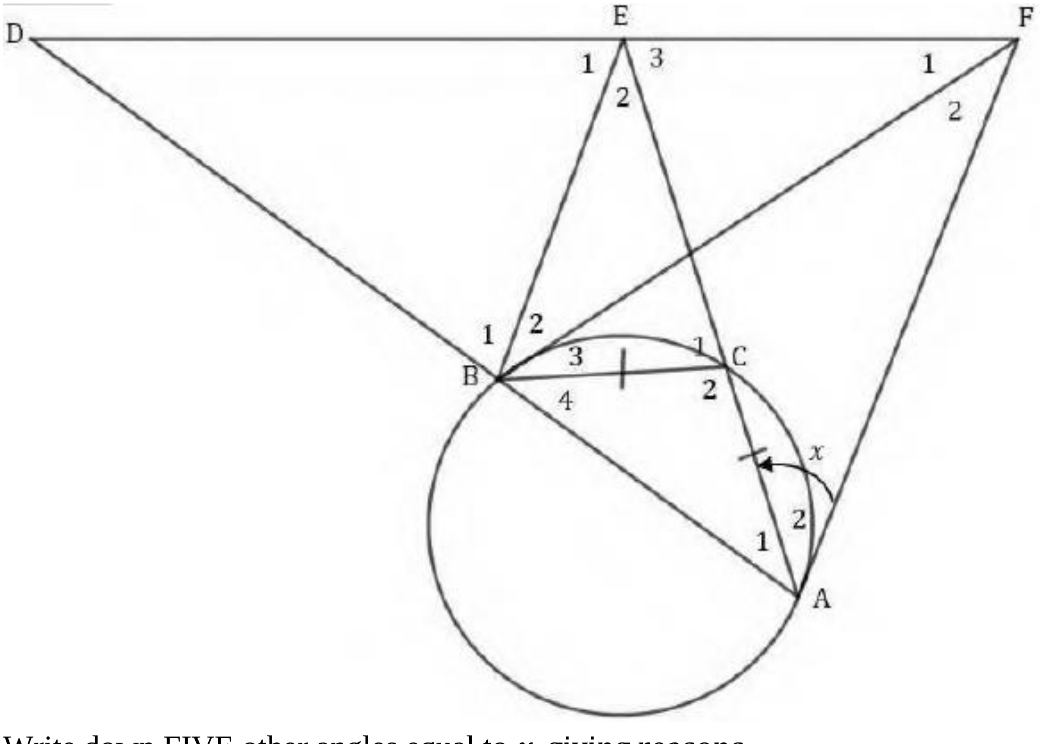
Activities

1. In the diagram below, a circle centred at O is drawn. H, J, G and L are points on the circle. ΔHJL is drawn. HOG bisects JL at M. HJ =$12\sqrt{5}$ units and JM = 12 units.



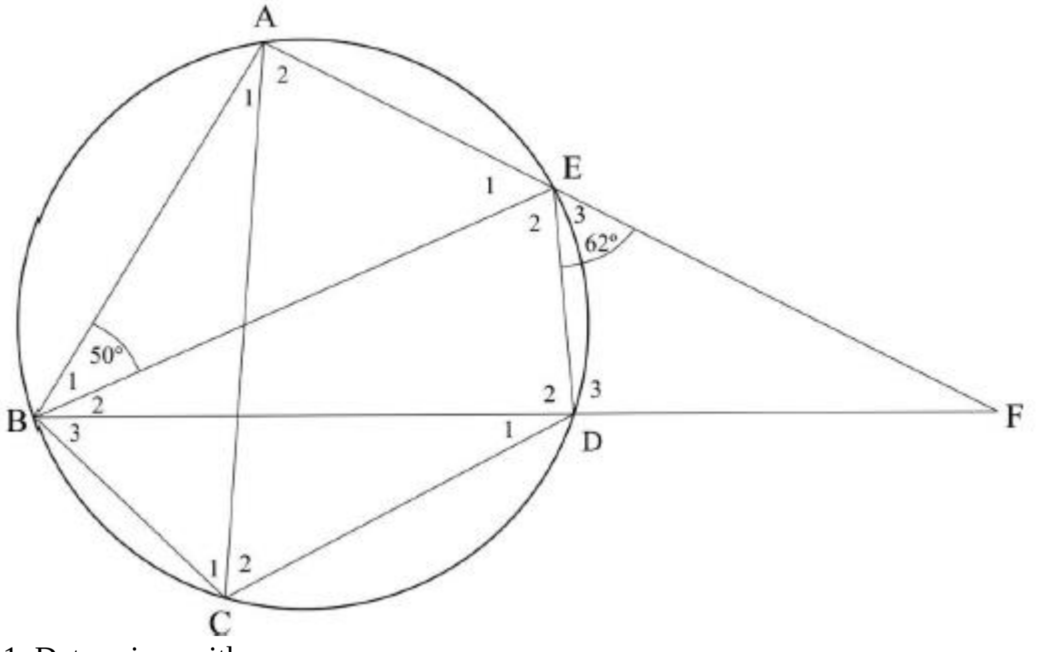
1.1 If MG = 6 units and OM = x, write HM in terms of x.
1.2 Hence, or otherwise giving reasons, the length of the radius of the circle.

2. FA and FB are tangents to the circle ABC with $BC = AC$. Chord AB is produced to D and chord AC is produced to meet DF at E. BC is joined. $FD \parallel CB$ and $C\hat{A}F = x$.



2.1 Write down FIVE other angles equal to x, giving reasons.
2.2 Hence deduce that:
   2.2.1 ABEF is a cyclic quadrilateral
   2.2.2 $AF = BD$

3. In the given diagram, A, B , C ,D and E are points on the circle. BE is a diameter. BD produced meets AE produced at F. $\widehat{E_3} = 62°$ and $\widehat{B_1} = 50°$



3.1 Determine, with reasons :

3.1.1 $B\hat{A}E$

3.1.2 $\widehat{E_1}$

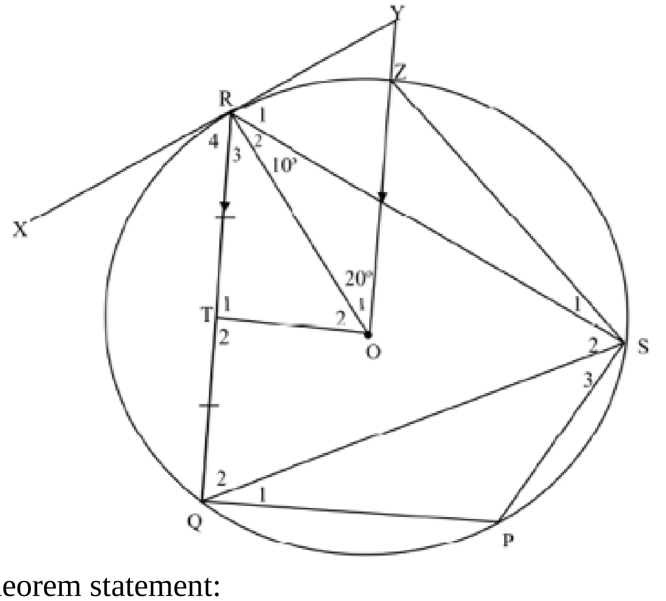
3.1.3 $\widehat{C_1}$

3.1.4 $\widehat{C_2}$

3.1.5 $A\hat{B}D$

4. In the diagram below, points P, Q, R and S are points on a circle with center O. OT bisects chord QR at T. XRY is a tangent to the circle at point R. OZ is produced to meet at Y where OY ||QR.

$R\hat{O}Y = 20°$ and $S\hat{R}O = 10°$. Chord SZ is drawn.



4.1 Complete the following theorem statement:

The angle between the tangent to a circle and the chord drawn from the point of contact is ….

4.2 Calculate, with reasons, the size of the following angles:

4.2.1 $\hat{S}_1$

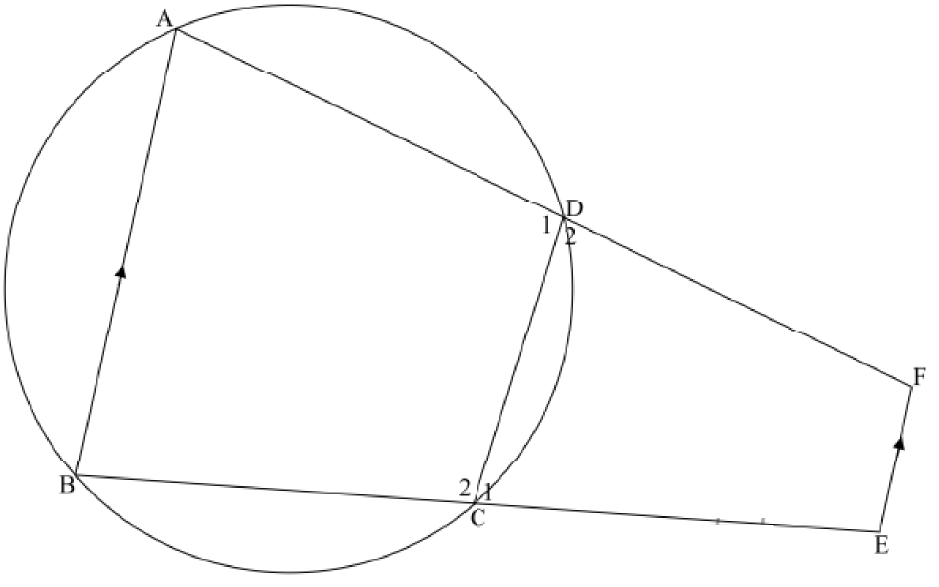
4.2.2 $\widehat{R_3}$

4.2.2 $\hat{P}$

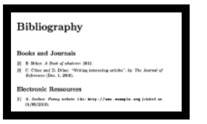
4.2.2 $\hat{S}_2$

4.3 Prove that XRY is a tangent to the circle passing through R, T and O.

5 In the diagram below, ABCD is a cyclic quadrilateral. Chords AD and BC are produced to F and E respectively. F and E are joined such that EF || AB.



Prove that CEFD is a cyclic quadrilateral.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3	CURRICULUM AND ASSESSMENT POLICY STATEMENT(CAPS) FET PHASE GRADES (10 – 12) MATHEMATICS
4	MATHEMATICS QUESTION PAPER GRADE 9 2013 EASTERN CAPE
5	MATHEMATICS QUESTION PAPER GRADE 9 2014 EASTERN CAPE
6	MATHEMATICS QUESTION PAPER GRADE 9 2016 EASTERN CAPE
7	MATHEMATICS QUESTION PAPER GRADE 9 2011 EASTERN CAPE
8	BISHOPS DIOCESAN COLLEGE GRADE 9 PAPER 2 2017
9	2021 Grade 12 MATHEMATICS EXAMINATION GUIDELINES
10	MATHEMATICS QUESTION PAPERS GRADE 12 (2008 – 2017) NATIONAL
11	MIND ACTION SERIES TEXT BOOK GRADE 12
12	MATHEMATICS QUESTION PAPER GRADE 9 2017 - 2020 GAUTENG
13	WWW.MATHISFUN.COM