JENN

Training and Consultancy

The path to enlightened education

SUBJECT: MATHEMATICS

ACTIVITY BOOK

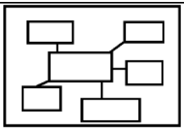
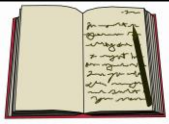
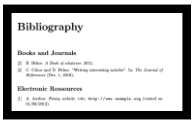
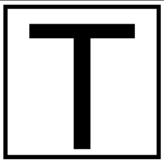
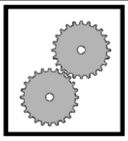
LEARNER

TERM 1

Patterns, Sequences and Series

CONTENTS **PAGE**

ICON DESCRIPTION

 **MIND MAP**	 **EXAMINATION GUIDELINE**	 **CONTENTS**	 **ACTIVITIES**
 **BIBLIOGRAPHY**	 **TERMINOLOGY**	 **WORKED EXAMPLES**	 **STEPS**

Formulae

Formulae to be used in this section are not limited to the following:

$T_n = a + (n-1)d$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$T_n = ar^{n-1}$

$S_n = \frac{a(r^n - 1)}{r - 1}$; $r \neq 1$

$S_\infty = \frac{a}{1-r}$; $-1 < r < 1$

$T_n = S_n - S_{n-1}$ (not provided in the information sheet)

$T_n = an^2 + bn + c$ (not provided in the information sheet)

$S_n = \frac{n}{2}(a + l)$ where "l" is the last term of the series

$N.B: l = T_n = a + (n-1)d$

(not provided in the information sheet)

TOPIC 1 : Quadratic Pattern

Outcomes: At the end of the session: The learner must

- know and use the general formula for quadratic patterns
- know and use the fact that the first differences of a quadratic sequence forms a linear pattern
- be able to extend the knowledge of quadratic patterns to quadratic functions

Activity 1

QUESTION 3

3.3 Consider the quadratic pattern: 5 ; 12 ; 29 ; 56 ; …

3.3.1 Write down the NEXT TWO terms of the pattern.

3.3.2 Prove that the first differences of this pattern will always be odd.

QUESTION 4

4.1 Consider the quadratic pattern: 3 ; 5 ; 8 ; 12 ; …

Determine the value of T_{26} .

4.2 A certain quadratic pattern has the following characteristics:

- $T_1 = p$
- $T_2 = 18$
- $T_4 = 4T_1$
- $T_3 - T_2 = 10$

Determine the value of p.

Activity 2

The number pattern 1, 5, 11, 19, … is such that the second difference is constant.

7.1 Determine the 5^{th} number in the pattern.

7.2 Derive a formula for the n^{th} number in the pattern.

7.3 What is the 100^{th} number in the pattern?

Activity 3

Consider the following sequence: 5; 12; 21; 32; ...

3.1 Write down the next term of the sequence.

3.2 Determine a formula for the n^{th} term of this sequence.

Activity 4

The following sequence of numbers forms a quadratic sequence:

$$-3\ ;\ -2\ ;\ -3\ ;\ -6\ ;\ -11\ ;\ \ldots$$

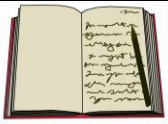
4.1 The first differences of the above sequence also form a sequence. Determine an expression for the general term of the first differences.

4.2 Calculate the first difference between the 35^{th} and 36^{th} terms of the quadratic sequence.

4.3 Determine an expression for the n^{th} term of the quadratic sequence.

4.4 Explain why the sequence of numbers will never contain a positive term.

Activity 5

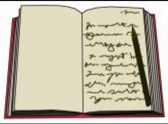


The sequence $4 \; ; \; 9 \; ; \; x \; ; \; 37; \ldots$ is a quadratic sequence.

3.1 Calculate x.

3.2 Hence, or otherwise, determine the n^{th} term of the sequence.

Activity 6

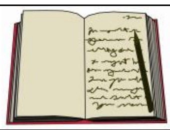


A quadratic pattern has a second term equal to 1, a third term equal to – 6 and a fifth term equal to – 14.

4.1 Calculate the second difference of this quadratic pattern.

4.2 Hence, or otherwise, calculate the first term of the pattern.

Activity 7



Consider the sequence: $6 \; ; \; 6 \; ; \; 2 \; ; \; -6 \; ; \; -18 \; ; \; \ldots$

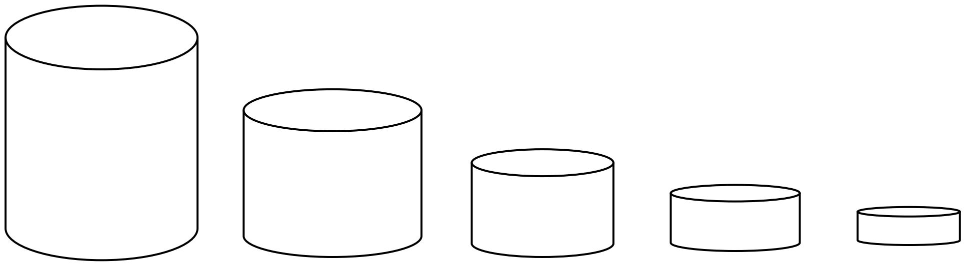
4.1 Write down the next term of the sequence, if the sequence behaves consistently.

4.2 Determine an expression for the n^{th} term, T_n.

4.3 Show that -6838 is in this sequence.

Activity 8

3.2 Twenty water tanks are decreasing in size in such a way that the volume of each tank is $\frac{1}{2}$ the volume of the previous tank. The first tank is empty, but the other 19 tanks are full of water.



Would it be possible for the first water tank to hold all the water from the other 19 tanks? Motivate your answer.

3.3 The n^{th} term of a sequence is given by $T_n = -2(n-5)^2 + 18$.

3.3.1 Write down the first THREE terms of the sequence.

3.3.2 Which term of the sequence will have the greatest value?

3.3.3 What is the second difference of this quadratic sequence?

3.3.4 Determine ALL values of n for which the terms of the sequence will be less than –110.

Activity 9

Consider the following sequence: 399 ; 360 ; 323 ; 288 ; 255 ; 224 ; …

2.1 Determine the n^{th} term T_n in terms of n.

2.2 Determine which term (or terms) has a value of 0.

2.3 Which term in the sequence will have the lowest value?

Activity 10

The sequence 3 ; 9 ; 17 ; 27 ; … is a quadratic sequence.

2.1 Write down the next term.

2.2 Determine an expression for the n^{th} term of the sequence.

2.3 What is the value of the first term of the sequence that is greater than 269?

Activity 11

An athlete runs along a straight road. His distance d from a fixed point P on the road is measured at different times, n, and has the form $d(n) = an^2 + bn + c$. The distances are recorded in the table below.

Time (in seconds)	1	2	3	4	5	6
Distance (in metres)	17	10	5	2	r	s

4.1 Determine the values of r and s.

4.2 Determine the values of a, b and c.

4.3 How far is the athlete from P when $n = 8$?

4.4 Show that the athlete is moving towards P when $n < 5$, and away from P when $n > 5$.

Activity 12

3.2 The arithmetic sequence 4 ; 10 ; 16 ; ... is the sequence of first differences of a quadratic sequence with a first term equal to 3.

Determine the 50$^{\text{th}}$ term of the quadratic sequence.

TOPIC 2 : Arithmetic Sequence and Series

Outcomes: At the end of the session: The learner must

- know and use the general formula for arithmetic sequence
- be able to derive and use the formular for the sum of arithmetic series
- be able to apply the knowledge of arithmetic sequence and series in real life situations

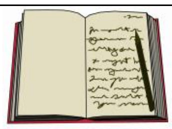
Activity 1

2.3 The following arithmetic sequence is given: 20 ; 23 ; 26 ; 29; … ; 101

2.3.1 How many terms are there in this sequence?

2.3.2 The even numbers are removed from the sequence.
Calculate the sum of the terms of the remaining sequence.

Activity 2



2.3 Prove that for any arithmetic sequence of which the first term is *a* and the constant difference is *d*, the sum to *n* terms can be expressed as $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$.

Activity 3

2.1 $3x + 1$; $2x$; $3x - 7$ are the first three terms of an arithmetic sequence. Calculate the value of x.

2.2 The first and second terms of an arithmetic sequence are 10 and 6 respectively.

2.2.1 Calculate the 11^{th} term of the sequence.

2.2.2 The sum of the first *n* terms of this sequence is –560. Calculate *n*.

Activity 4

The following is an arithmetic sequence:

$$1-p \;\; ; \;\; 2p-3 \;\; ; \;\; p+5 \;\; ; \;\; \ldots$$

3.1 Calculate the value of p.

3.2 Write down the value of:

3.2.1 The first term of the sequence

3.2.2 The common difference

3.3 Explain why none of the numbers in this arithmetic sequence are perfect squares.

Activity 5

Matli's annual salary is R120 000 and his expenses total R90 000. His salary increases by R12 000 each year while his expenses increase by R15 000 each year. Each year he saves the excess of his income.

4.1 Represent his total savings as a series.

4.2 If Matli continues to manage his finances this way, after how many years will he have nothing left to save?

4.3 Matli calculates that if his expenses increase by x rand every year (instead of R15 000 each year), he will spend as much as he earns in the 25^{th} year. Determine x.

Activity 6

Given the arithmetic series: $-7-3+1+\ldots+173$

2.1 How many terms are there in the series?

2.2 Calculate the sum of the series.

Activity 7

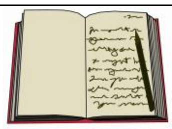
2.2 Consider the arithmetic sequence: $-8 ; -2 ; 4 ; 10 ; ...$

2.2.1 Write down the next term of the sequence.

2.2.2 If the n^{th} term of the sequence is 148, determine the value of n.

2.2.3 Calculate the smallest value of n for which the sum of the first n terms of the sequence will be greater than 10 140.

Activity 8



3.1 Given the arithmetic sequence: $w-3 \; ; \; 2w-4 \; ; \; 23-w$

3.1.1 Determine the value of w.

3.1.2 Write down the common difference of this sequence.

3.2 The arithmetic sequence 4 ; 10 ; 16 ; ... is the sequence of first differences of a quadratic sequence with a first term equal to 3.

Determine the 50^{th} term of the quadratic sequence.

Activity 9

A cyclist, preparing for an ultra cycling race, cycled 20 km on the first day of training. She increases her distance by 4 km every day.

2.1 On which day does she cycle 100 km?

2.2 Determine the total distance she would have cycled from day 1 to day 14.

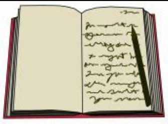
2.3 Would she be able to keep up this daily rate of increase in distance covered indefinitely? Substantiate your answer.

TOPIC 3 : Geometric Sequence and Series, and Sum to infinity

Outcomes: At the end of the session: The learner must

- know and use the general formula for geometric sequence
- be able to derive and use the formular for the sum of geometric series
- be able to apply the knowledge of geometric sequence and series in real life situations
- know that sum to infinity exists for a convergent (-1 < r < 1) geometric series

Activity 1



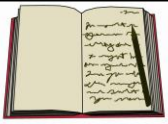
Given the geometric series: $8x^2 + 4x^3 + 2x^4 + ...$

3.1 Determine the n^{th} term of the series.

3.2 For what value(s) of x will the series converge?

3.3 Calculate the sum of the series to infinity if $x = \frac{3}{2}$.

Activity 2



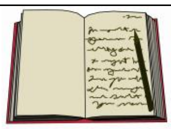
4.1 Prove that $a + ar + ar^2 + ...\text{ (to } n \text{ terms)} = \frac{a(1-r^n)}{1-r}$ for $r \neq 1$

4.2 Given the geometric series $15 + 5 + \frac{5}{3} + ...$

4.2.1 Explain why the series converges.

4.2.2 Evaluate $\sum_{n=1}^{\infty} 5(3^{2-n})$

Activity 3



2.2 The following sequence forms a convergent geometric sequence: $5x\ ;\ x^2\ ;\ \frac{x^3}{5}\ ;\ ...$

2.2.1 Determine the possible values of x.

2.2.2 If $x = 2$, calculate S_∞.

Activity 4

Data regarding the growth of a certain tree has shown that the tree grows to a height of 150 cm after one year. The data further reveals that during the next year, the height increases by 18 cm. In each successive year, the height increases by $\frac{8}{9}$ of the previous year's increase in height. The table below is a summary of the growth of the tree up to the end of the fourth year.

	First year	Second year	Third year	Fourth year
Tree height (cm)	150	168	184	$198\frac{2}{9}$
Growth (cm)		18	16	$14\frac{2}{9}$

5.1 Determine the increase in the height of the tree during the seventeenth year.

5.2 Calculate the height of the tree after 10 years.

5.3 Show that the tree will never reach a height of more than 312 cm.

Activity 5

3.1 Given the geometric sequence: 27 ; 9 ; 3 …

3.1.1 Determine a formula for T_n, the n^{th} term of the sequence.

3.1.2 Why does the sum to infinity for this sequence exist?

3.1.3 Determine S_∞.

Activity 6

3.2 Given the geometric series: $3+1+\frac{1}{3}+\ldots$

Calculate the sum to infinity.

Activity 7

A sequence of squares, each having side 1, is drawn as shown below. The first square is shaded, and the length of the side of each shaded square is half the length of the side of the shaded square in the previous diagram.

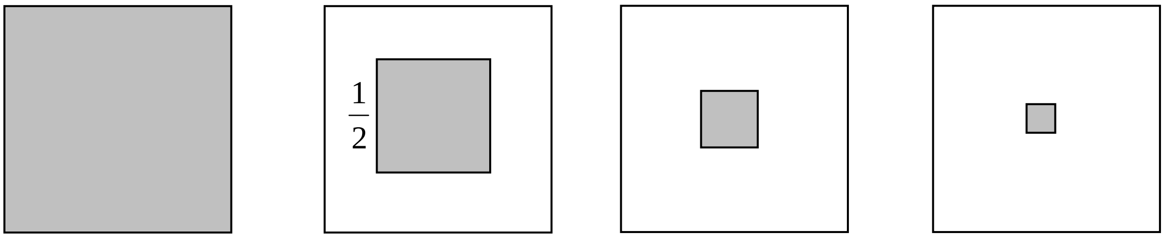


DIAGRAM 1 **DIAGRAM 2** **DIAGRAM 3** **DIAGRAM 4**

5.1 Determine the area of the unshaded region in DIAGRAM 3.

5.2 What is the sum of the areas of the unshaded regions on the first seven squares?

Activity 8

3.1 The first two terms of an infinite geometric sequence are 8 and $\frac{8}{\sqrt{2}}$. Prove, without the use of a calculator, that the sum of the series to infinity is $16+8\sqrt{2}$.

Activity 9

2.1 A geometric sequence has $T_3 = 20$ and $T_4 = 40$.

Determine:

2.1.1 The common ratio

2.1.2 A formula for T_n

Activity 10

3.1 Consider the geometric sequence: $4\ ;\ -2\ ;\ 1\ \ldots$

3.1.1 Determine the next term of the sequence.

3.1.2 Determine n if the n^{th} term is $\frac{1}{64}$.

3.1.3 Calculate the sum to infinity of the series $4 - 2 + 1\ \ldots$

3.2 If x is a REAL number, show that the following sequence can NOT be geometric:

$$1\ ;\ x+1\ ;\ x-3\ \ldots$$

Activity 11

2.1 Given the geometric series: $256 + p + 64 - 32 + \ldots$

2.1.1 Determine the value of p.

2.1.2 Calculate the sum of the first 8 terms of the series.

2.1.3 Why does the sum to infinity for this series exist?

2.1.4 Calculate S_∞

Activity 12

In a geometric series, the sum of the first n terms is given by $S_n = p\left(1-\left(\frac{1}{2}\right)^n\right)$ and the sum to infinity of this series is 10.

4.1 Calculate the value of p.

4.2 Calculate the second term of the series.

TOPIC 4 : Sigma Notation

Outcomes: At the end of the session: The learner must

- ba able to use the sigma notation for both arithmetic and geometric series

Activity 1

Given: $\sum_{t=0}^{99}(3t-1)$

3.1 Write down the first THREE terms of the series.

3.2 Calculate the sum of the series.

Activity 2

2.1 Evaluate: $\sum_{n=1}^{20} 3^{n-2}$

Activity 3

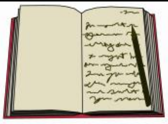
2.2 Determine the value of P if $P = \sum_{k=1}^{13} 3^{k-5}$

Activity 4

3.2 The following geometric series is given: $x = 5 + 15 + 45 + \ldots$ to 20 terms.

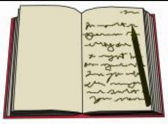
3.2.1 Write the series in sigma notation.

3.2.2 Calculate the value of x.

Activity 5	

Given the arithmetic series: $-7 - 3 + 1 + \ldots + 173$

2.3 Write the series in sigma notation.

Activity 6 Source : Grade 12 Feb-Mar 2013 Question 2	

2.3 Calculate $\sum_{k=1}^{30}(3k+5)$

TOPIC 5 : Mixed Patterns

Outcomes: At the end of the session: The learner must

- be able to successfully work through questions that are having mixed sequences
- know when to use the formula $T_n = S_n - S_{n-1}$

Activity 1

2.1 Consider the sequence: $\frac{1}{2}; 4; \frac{1}{4}; 7; \frac{1}{8}; 10; \ldots$

2.1.1 If the pattern continues in the same way, write down the next TWO terms in the sequence.

2.1.2 Calculate the sum of the first 50 terms of the sequence.

Activity 2

4.3 The sum of the first n terms of a sequence is given by $S_n = 2^{n+2} - 4$

4.3.1 Determine the sum of the first 24 terms.

4.3.2 Determine the 24^{th} term.

4.3.3 Prove that the n^{th} term of the sequence is 2^{n+1}.

Activity 3

2.1 Tebogo and Matthew's teacher has asked that they use their own rule to construct a sequence of numbers, starting with 5. The sequences that they have constructed are given below.

Matthew's sequence: 5 ; 9 ; 13 ; 17 ; 21 ; …
Tebogo's sequence: 5 ; 125 ; 3 125 ; 78 125 ; 1 953 125 ; …

Write down the n^{th} term (or the rule in terms of n) of:

2.1.1 Matthew's sequence

2.1.2 Tebogo's sequence

2.2 Nomsa generates a sequence which is both arithmetic and geometric. The first term is 1. She claims that there is only one such sequence. Is that correct? Show ALL your workings to justify your answer.

Activity 4

The following sequence is a combination of an arithmetic and a geometric sequence:

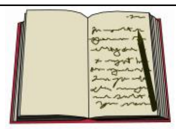
$$3 \; ; 3 \; ; 9 \; ; 6 \; ; 15 \; ; 12 \; ; \ldots$$

3.1 Write down the next TWO terms.

3.2 Calculate $T_{52} - T_{51}$.

3.3 Prove that ALL the terms of this infinite sequence will be divisible by 3.

Activity 5



Consider the sequence: 3 ; 9 ; 27 ; ...

Jacob says that the fourth term of the sequence is 81.
Vusi disagrees and says that the fourth term of the sequence is 57.

3.1 Explain why Jacob and Vusi could both be correct.

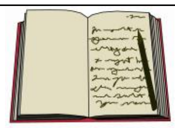
3.2 Jacob and Vusi continue with their number patterns.

Determine a formula for the n^{th} term of:

3.2.1 Jacob's sequence

3.2.2 Vusi's sequence

Activity 6



2.2 The following sequence has the property that the sequence of numerators is arithmetic and the sequence of denominators is geometric:

$$\frac{2}{1} \; ; \frac{-1}{5} \; ; \frac{-4}{25} \; ; \ldots$$

2.2.1 Write down the FOURTH term of the sequence.

2.2.2 Determine a formula for the n^{th} term.

2.2.3 Determine the 500^{th} term of the sequence.

2.2.4 Which will be the first term of the sequence to have a NUMERATOR which is less than –59?

EXCTRACTS FROM PAST PAPERS

ACTIVITY 1

Given the geometric series: $x + 90 + 81 + \ldots$

1.1 Calculate the value of x.

1.2 Show that the sum of the first n terms is $S_n = 1\,000\left(1 - (0,9)^n\right)$.

1.3 Hence, or otherwise, calculate the sum to infinity.

ACTIVITY 2

QUESTION 3

Consider the quadratic number pattern: $-145\ ; -122\ ; -101\ ; \ldots$

3.1 Write down the value of T_4.

3.2 Show that the general term of this number pattern is $T_n = -n^2 + 26n - 170$.

3.3 Between which TWO terms of the quadratic number pattern will there be a difference of -121?

3.4 What value must be added to each term in the number pattern so that the value of the maximum term in the new number pattern formed will be 1?

ACTIVITY 3

QUESTION 4

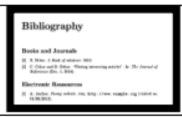
Consider the linear pattern: $5\ ; 7\ ; 9\ ; \ldots$

4.1 Determine T_{51}.

4.2 Calculate the sum of the first 51 terms.

4.3 Write down the expansion of $\sum_{n=1}^{5000}(2n+3)$. Show only the first 3 terms and the last term of the expansion.

4.4 Hence, or otherwise, calculate $\sum_{n=1}^{5000}(2n+3) + \sum_{n=1}^{4999}(-2n-1)$.
ALL working details must be shown.

Bibliography	

1. Grade 11 November 2013 Eastern Cape Question paper
2. Grade 11 November 2015 Question paper
3. Grade 12 November 2008 – 2012 Question papers
4. Grade 12 Feb – Mar 2009 – 2014 Question papers

Outcomes reached

Add outcomes set	**YES**	**NO**
Topic 1 • know and use the general formula for quadratic patterns • know and use the fact that the first differences of a quadratic sequence forms a linear pattern • be able to extend the knowledge of quadratic patterns to quadratic functions		
Topic 2 • know and use the general formula for arithmetic sequence • be able to derive and use the formular for the sum of arithmetic series • be able to apply the knowledge of arithmetic sequence and series in real life situations		
Topic 3 • know and use the general formula for geometric sequence • be able to derive and use the formular for the sum of geometric series • be able to apply the knowledge of geometric sequence and series in real life situations • know that sum to infinity exists for a convergent (-1 < r < 1) geometric series		
Topic 4 • ba able to use the sigma notation for both arithmetic and geometric series		
Topic 5 • be able to successfully work through questions that are having mixed sequences • know when to use the formula $T_n = S_n - S_{n-1}$		